A
TREATISE
ON THE
DIFFERENTIAL
AND
INTEGRAL CALCULUS.

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SECTION I.

DEFINITIONS AND FIRST PRINCIPLES.

(1.) In the Differential Calculus, numbers or quantities are considered as being constant or variable; those whose values do not change during any investigation, whether they are known or not, being called *constants;* while those whose values change, or are conceived to be altered, are called *variables.* Constants are generally represented by the first letters of the alphabet, and variables by the last letters. Thus in $ax + b$, $y = ax^2 + bx + c$; a, b, c are constants, and x, y are variables.

(2.) Variables that are entirely arbitrary, or arbitrary within certain limits, are called *independent variables;* while those variables whose values depend on the values of one or more others that are independent of them, are called *functions of the variables*, on whose values they depend. When the dependence of a variable on one or more others is expressed or given, the variable is called an *explicit function* of the variables on whose values it depends; but if the manner in which a variable depends on one or more others is neither expressed nor known, and is to be found from the solution of one or more equations or in any other way, the variable is called an *implicit function* of the variables on whose values its value depends. It may be added, that a variable which is expressed in variables and constants, is not considered as being a function of the constants. Thus, in $y = 3x + 7$, $y = ax + b$, y is an explicit function of x, and x is an implicit of y; and neither y nor x is considered as being a function of the figures 3, 7, or of the constants a, b.

To signify in a general way, that any variable, as y, is an explicit function of another variable, as x, we write them in such forms as $y = \mathrm{F}(x)$, $y = f(x)$, $y = \varphi(x)$, $y = \psi(x)$, &c., either of which is read by saying that y is an explicit function of x: and to show that y is an implicit function of x, we use such forms as $\mathrm{F}(x, y) = 0$, $f(x, y) = 0$, $\varphi(x, y) = 0$, &c., which are read by saying that y is an implicit function of x.

(3.) When a function of a variable and the variable increase or decrease together, the function is sometimes called *an increasing function;* but if the function increases when the variable decreases, or the reverse, the function is said to be a *decreasing function.* Thus, in $y = ax + b$, y is an increasing function of x; and in $y = \frac{a}{x}$, y is a decreasing function of x.

(4.) If x represents any arbitrary variable, which is changed into x'; then $x' - x$, the difference of the values of x (found by subtracting the first from the second), is called (in the Differential Calculus), the differential of x (x being the first value of the variable), and is expressed by dx, by writing the small letter d (the first letter in the word differential) before or to the left of the variable x.

If X stands for any function of x, *and the algebraic sum of all the changes in the value of* X, that result from the separate variation $x' - x = dx$, *of each x is taken, it will equal* (*what is called*) the differential of X; which is expressed by

dX, as in the case of x. If dX is divided by dx, the quotient is called the differential coefficient of dx (the differential of the independent variable x), since it is the coefficient of dx in $dX = \frac{dX}{dx} dx$. Because constants do not change their values, it is clear that the differential of x or X when increased or diminished by any constant, will be dx or dX, the same as before. And if x or X has a constant factor or divisor, then dx or dX, when multiplied or divided by the constant, will be the corresponding differential.

Thus, if $X = x^2 = xx$, then if X′ represents the value of X when either x (in xx) is changed into x', it is clear that from the change in the first x we shall get $X' = x'x$, or subtracting $X = xx$, we get $X' - X = x'x - xx = x(x' - x)$; and from $X = xx$, by changing the second x into x', we shall in like manner get $X' - X = xx' - xx = x(x' - x)$; consequently, from the addition of these expressions, we get $2(X' - X) = 2x(x' - x)$.

Because $2x(x' - x)$ is clearly the whole change that can take place in $x^2 = xx$, according to the preceding principles; it is clear (from the definition of the differential of a function of a variable) that for $2(X' - X)$ we must put dX the differential of X, and since $x' - x = dx$, the preceding equation becomes $dX = 2xdx$, which expresses the differential of $X = x^2$; and dividing by dx (the differential of the independent variable x), we have $\frac{dX}{dx} = 2x$, for the differential coefficient of $X = x^2$. If X equals x^3 or x^4 or . . . . x^n, n being a positive whole number, we shall, in like manner, get $dX = 3x^2dx$ or $4x^3dx$ or . . . . $nx^{n-1}dx$, for their differentials, and $\frac{dX}{dx} = 3x^2$ or $4x^3$ or . . . . nx^{n-1} for their differential coefficients.

Similarly, if $X = ax + b$, or $a'x^2 + b'$, or $a''x^3 + b''$, &c., we shall get $dX = adx$ or $2a'xdx$ or $3a''x^2dx$, &c., for their differentials, and $\frac{dX}{dx} = a$ or $2a'x$ or $3a''x^2$ or, &c., for their differential coefficients; and generally, m being a positive integer, the differential and differential coefficient of $X = (Ax^m + B) \div C$, will be expressed by $dX = mAx^{m-1}dx \div C$, and $\frac{dX}{dx} = mAx^{m-1} \div C$: which results clearly follow from the consideration that A, B, C, do not change their values, or that their differentials equal naught.

We are now prepared to find the differential of a variable, or function of one or more variables, when it is affected by any given exponent; or, as is sometimes said, we are prepared to find the differential of any given power or root of a variable or function.

1. Let it be proposed to find the differential of $X = x^{\frac{n}{m}}$ supposing m and n to be positive integers. Since the equation is equivalent to $X^m = x^n$, which is the same as $(x^{\frac{n}{m}})^m = x^n$ an identical equation; by taking their differentials (according to what has been shown), we shall clearly have $mX^{m-1}dX = nx^{n-1}dx$; consequently, since $X^{m-1} = x^{\frac{n(m-1)}{m}} = x^{n-\frac{n}{m}}$, we shall have $dX = \frac{n}{m}x^{\frac{n}{m}-1}dx$, as required.

2. Let $X = x^{-\frac{n}{m}}$, or $Xx^{\frac{n}{m}} = 1$, be proposed, in order to find dX, the differential of $X = x^{-\frac{n}{m}}$, supposing as before m and n to be any positive integers. Because $Xx^{\frac{n}{m}} = 1$, is essentially the same as the identical equation $x^{-\frac{n}{m}}x^{\frac{n}{m}} = 1$, it is clear that the differential of Xx^m must equal naught, since the differential of its equivalent, 1, equals naught.

It is clear (from the nature of a differential), that in finding the differential of $Xx^{\frac{n}{m}}$, we may take the differential of each factor regarding the other as constant, and add the results for the whole differential; consequently we shall have

$$Xdx^{\frac{n}{m}} + x^{\frac{n}{m}}dX = 0, \text{ or } dX = -\frac{n}{m}Xx^{-1}dx = -\frac{n}{m}x^{-\frac{n}{m}-1}dx,$$

as required.

3. If $X = \left(a^{\pm\frac{p}{q}} \pm x^{\pm\frac{p}{q}}\right)^{\pm\frac{n}{m}}$, we shall clearly, as before, have $dX = \pm\frac{n}{m}\frac{p}{q}\left(a^{\pm\frac{p}{q}} \pm x^{\pm\frac{p}{q}}\right)^{\pm\frac{n}{m}-1} x^{\pm\frac{p}{q}-1}dx$, for its differential.

4. Hence, the differential of any given power, or root of a variable or function, can be found by the following

RULE.

Multiply the power or root by its index, subtract 1 *or unity from the index, in the product; then, multiply the result by the differential of the variable or function, for the required differential.*

EXAMPLES.

1. To find the differentials of x^5 and $(x^m)^n$.

Here we have the variable x raised to the 5th power, and the function x^m raised to the nth power, the indices of the powers being 5 and n; consequently, by the rule, we shall have $5x^{5-1}dx = 5x^4dx$ and

$$n\,(x^m)^{n-1}dx^m = nx^{mn-m} \times mx^{m-1}dx = mnx^{mn-1}dx$$

for their differentials: noticing, that the second differential is manifestly correct, since $(x^m)^n = x^{mn}$.

2. To find the differentials of $\sqrt{x} = x^{\frac{1}{2}}$ and $\sqrt[3]{x^2} = x^{\frac{2}{3}}$.

Here $\frac{1}{2}$ and $\frac{2}{3}$ are the indices, and by the rule we shall have

$d\sqrt{x} = dx^{\frac{1}{2}} = \frac{1}{2}x^{\frac{1}{2}-1}dx = \frac{1}{2}x^{-\frac{1}{2}}dx = \frac{1}{2\sqrt{x}}dx = \frac{1}{2x^{\frac{1}{2}}}dx$; and

$d\sqrt[3]{x^2} = \frac{2}{3}\,\frac{1}{\sqrt[3]{x}}dx = \frac{2}{3}\,\frac{1}{x^{\frac{1}{3}}}\,dx$; for the differentials.

3. The differentials of $7y^6$ and $5z^{\frac{3}{5}}$, are $42y^5dy$ and $3z^{-\frac{2}{5}}dz$; which are obtained by multiplying the differentials of y^6 and $z^{\frac{3}{5}}$, by their coefficients 7 and 5, as we clearly ought to do.

4. The differentials of $ax^m \pm b$ and $\frac{a}{b}x^n \pm c$, are

$$max^{m-1}dx \text{ and } \frac{na}{b}x^{n-1}dx,$$

which are clearly correct, since the constants connected with the variable parts by $\pm$, must clearly disappear when the differentials are taken, and that the differentials of ax^m and $\frac{a}{b}x^n$ must evidently be a and $\frac{a}{b}$ times the differentials of x^m and x^n.

5. The differentials of $2\sqrt{(a^2+x^2)} = 2\,(a^2+x^2)^{\frac{1}{2}}$, and $\frac{3}{5}\sqrt[3]{(a^2+x^2)} = \frac{3}{5}(a^2+x^2)^{\frac{1}{3}}$, are $2\,(a^2+x^2)^{-\frac{1}{2}}xdx = \frac{2xdx}{\sqrt{(a^2+x^2)}}$ and $\frac{2}{5}\,(a^2+x^2)^{-\frac{2}{3}}xdx = \frac{2xdx}{5\,(a^2+x^2)^{\frac{2}{3}}}$

6. The differentials of $(a^2+x^2)^{-2}$ and $(a^2-x^2)^{-3}$, are $-4\,(a^2+x^2)^{-3}xdx$ and $6\,(a^2-x^2)^{-4}xdx$.

7. The differentials of $(a^2+3x^2)^{-7}$ and $(a^2-3x^2)^{-\frac{7}{6}}$, are $-42\,(a^2+3x^2)^{-8}xdx = -\frac{42xdx}{(a^2+3x^2)^8}$ and $\frac{7xdx}{(a^2-3x^2)^{\frac{13}{6}}}$.

8. The differentials of $(a^2+x^{-2})^{-2}$ and $(a^2-x^{-2})^{-2}$, are $\frac{4x^{-3}dx}{(a^2+x^{-2})^3} = \frac{4x^3dx}{(a^2x^2+1)^3}$, and $-\frac{4x^3dx}{(a^2x^2-1)^3}$.

9. The differentials of $(2y^2+3x^2)^2$ and $(2y^2-3x^2)^2$, are $(4y^2+6x^2)\,(4ydy+6xdx)$ and $2\,(2y^2-3x^2)\,(4ydy-6xdx)$.

(5.) If X is a function of any number of variables that are independent of each other, *it is customary to call the differential of* X *taken with respect to any one of the independent variables, a partial differential of* X, *and the corresponding differential coefficient is also called a partial differential coefficient; and the algebraic sum of all the partial differentials of* X, *is called its total differential.*

If X has two or more terms that are functions of the same variable, it is clear that we may find the differentials of such terms as before, and then take the algebraic sum of the differentials for the differential of the sum of such terms.

Thus, if X is a function of x, y, z, &c., we shall have $\frac{dX}{dx}dx$, $\frac{dX}{dy}dy$, $\frac{dX}{dz}dz$, &c., for the partial differentials of X, whose sum gives $dX = \frac{dX}{dx}dx + \frac{dX}{dy}dy + \frac{dX}{dz}dz +$, &c.; for the complete or total differential of X; and $\frac{dX}{dx}$, $\frac{dX}{dy}$, $\frac{dX}{dz}$, &c., are the partial differential coefficients. And if we have $X = 3ax^2 - bx + c$, by taking the differentials of its terms separately we shall have $6axdx$ and $-bdx$ for the partial differentials, whose sum gives $dX = 6axdx - bdx = (6ax - b)dx$ for the complete or total differential of the proposed expression; and, of course, $\frac{dX}{dx} = 6ax = b$ is the corresponding differential coefficient.

Remarks.—1. If X is a function of a single variable, its differential coefficient is sometimes indicated by writing the capital D before or to the left of X: thus, DX signifies that the differential coefficient of X is to be taken; as in

$D(ax^3 - bx + c) = 3ax^2 - b$, called the first derived function of $ax^3 - bx + c$. And if X is a function of x, y, &c., the partial differential coefficients $\frac{dX}{dx}$, $\frac{dX}{dy}$, &c., are sometimes expressed by the forms D_xX, D_yX, &c.

2. To indicate that the differential of a compound quantity is to be taken, we put it under a vinculum or inclose it in a parenthesis, to which we prefix $d.$ or d (called the characteristic of differentials), and when the differential has been found, the quantity is said to have been differentiated. Thus $d.(x^2+y^2 - az)$ or $d\,(x^2 + y^2 - az)$ indicates the differential of $x^2 + y^2 - az$, which being taken, gives $d\,(x^2 + y^2 - az) = 2xdx + 2ydy - adz.$

To make what has been done more evident, take the following

EXAMPLES.

1. To find the differential and differential coefficients of $X = 3x^2 - 5y^2 + 9z^3$.

Here $dX = 6xdx - 10ydy + 27z^2dz$; and $\frac{dX}{dx} = 6x$, $\frac{dX}{dy} = -10y$, and $\frac{dX}{dz} = 27z^2$, are the differential coefficients.

2. Perform what is expressed by $d(\sqrt{x^2 - 2y^2} + az)$ and $d\,(x^3 - x^2 + x - \overline{3y^4 - 9y^3 + 7})$.

The answers are $\frac{xdx - 2ydy}{\sqrt{(x^2 - 2y^2)}} + adz$, and

$$3x^2dx - 2xdx + dx - 12y^3dy + 27y^2dy;$$

and the partial differential coefficients are

$$\frac{x}{\sqrt{(x^2 - 2y^2)}},\ -\frac{2y}{\sqrt{(x^2 - 2y^2)}}, \text{ and } 3x^2 - 2x + 1,\ -12y^3 + 27y^2.$$

3. Perform what is indicated by

$$Dy\,(x^3 - 3y^5) \text{ and } Dx_{,}y\,(ax^4 - y^5 + z).$$

Ans. $-15y^4$, $4ax^3$, and $-5y^4$; when $Dx_{,}y$ is used to indicate the differential coefficients with reference to x and y.

4. To find the differential of the product of any number of factors, as X, Y, Z, &c.; which may (if required) be functions of any variables.

Here it is easy to perceive (from the nature of differentials) that $d(XY) = XdY + YdX$,

$$d\,(XYZ) = XYdz + XZdY + YZdX, \text{ \&c.},$$

which are of like forms, are the sought answers.

(6.) It clearly follows from the preceding example, that the differential of a product can be found by the following

RULE.

The differential of the product of any number of variables or functions, *equals the (algebraic) sum of the differentials, that result from the differential of each factor multiplied by the product of all the remaining factors.*

EXAMPLES.

1. The differentials of xy and $3x^2y$, are $xdy + ydx$ and $3\,(x^2dy + 2yxdx) = 3x^2dy + 6yxdx$.

2. The differentials of xx^2 and x^3x^4, equal

$2x^2dx + x^2dx = 3x^2dx$, and $4x^3x^3dx + 3x^4x^2dx = 7x^6dx$; which are clearly the same as the differentials of x^3 and x^7, as they ought to be.

3. The differential of $(x^2 + y^2)\,(x^2 - y^2)$, is

$$(x^2 + y^2)\,(2xdx - 2ydy) + (x^2 - y^2)\,(2xdx + 2ydy)$$
$$= 4\,(x^3dx - y^3dy).$$

4. The differentials of $\sqrt{(a^2 + x^2)} \times \sqrt{(a^2 - x^2)}$, and $2x^{\frac{3}{2}}x^{\frac{5}{2}}$ are

$$\sqrt{a^2 + x^2} \times -\frac{xdx}{\sqrt{a^2 - x^2}} + \sqrt{(a^2 - x^2)} \times \frac{xdx}{\sqrt{a^2 + x^2}}$$

$$= -\frac{2x^3dx}{\sqrt{(a^4 - x^4)}}$$

and
$$5x^{\frac{3}{2}}x^{\frac{3}{2}}\,dx + 3x^{\frac{5}{2}}x^{\frac{1}{2}}dx = 8x^3dx.$$

5. The differentials of axy^2z^3 and $\frac{b}{c}x^2y^{-2}z^{-3}$, are

$$a\,(3xy^2z^2dz + 2xz^3ydy + y^2z^3dx)$$

and
$$\frac{b}{c}\,(-\,3x^2y^{-2}z^{-4}dz - 2x^2z^{-3}y^{-3}dy + 2y^{-2}z^{-3}xdx).$$

6. The differentials of $\frac{x}{y} = xy^{-1}$ and $\frac{x^2}{y^3} = x^2y^{-3}$, are

$$d\frac{x}{y} = -\,xy^{-2}dy + y^{-1}dx = \frac{ydx - xdy}{y^2}$$

and
$$d\frac{x^2}{y^3} = 2xy^{-3}dx - 3x^2y^{-4}dy.$$

Remark.—If we put $X = \frac{x}{y}$, we shall have $Xy = x$, whose differential gives $Xdy + ydX = dx$; or, since $X = \frac{x}{y}$, we have $\frac{x}{y}\,dy + yd\,\frac{x}{y} = dx$; consequently, we shall have $d\frac{x}{y} = \frac{ydx - xdy}{y^2}$, which is the same as found from xy^{-1}.

(7.) It follows from the preceding example and the remark, that the differential of a fraction can be found from the following

RULE.

Multiply the denominator by the differential of the numerator, and from the product subtract the numerator mul-

tiplied by the differential of the denominator, and divide the remainder by the square of the denominator.

EXAMPLES.

1. The differentials of $\frac{x^2}{x}$ and $\frac{x}{x^2}$ are

$$\frac{2x^2dx - x^2dx}{x^2} = \frac{x^2dx}{x^2} = dx \quad \text{and} \quad \frac{x^2dx - 2x^2dx}{x^4} = -\frac{dx}{x^2},$$

which are clearly correct; since the form

$$\frac{x^2}{x} = x, \text{ and } \frac{x}{x^2} = x^{-1}.$$

2. The differentials of $\left(\frac{x}{y}\right)^n$ and $\frac{x^n}{y^m}$, are

$$n\left(\frac{x}{y}\right)^{n-1} d\left(\frac{x}{y}\right) = n\left(\frac{x}{y}\right)^{n-1} \frac{ydx - xdy}{y^2}$$

and
$$\frac{ny^m x^{n-1}dx - mx^n y^{m-1}dy}{y^{2m}}$$

3. The differentials of $\frac{x-y}{x+y}$ and $\frac{x^2-y^2}{x^2+y^2}$, are $\frac{2(ydx - xdy)}{(x+y)^2}$ and $4xy\frac{(ydx - xdy)}{(x^2+y^2)^2}$.

4. The differentials of $\frac{yz}{x}$ and of $\frac{a^n \pm x^n}{x^n} = \frac{a^n}{x^n} \pm 1$, are

$$\frac{x(ydz + zdy) - yzdx}{x^2} \text{ and } \frac{x^n}{x^{2n}} d(a^n \pm x^n) - (a^n \pm x^n)\, dx^n =$$

$$-\frac{a^n dx^n}{x^{2n}} = -\frac{na^n x^{n-1}dx}{x^{2n}} = -\frac{na^n dx}{x^{n+1}} = d\frac{a^n}{x^n} = d(a^n x^{-n}).$$

5. The differentials of $\frac{a}{a+x}$ and $\frac{a}{a-x}$, are $-\frac{adx}{(a+x)^2}$ and $\frac{adx}{(a-x)^2}$.

6. The differentials of $\frac{x}{\sqrt{(a^2+x^2)}}=\frac{x}{(a^2+x^2)^{\frac{1}{2}}}$ and

$$\frac{x}{\sqrt{(a^2-x^2)}}=\frac{x}{(a^2-x^2)^{\frac{1}{2}}}, \text{ are}$$

$$\frac{dx}{\sqrt{(a^2-x^2)}}-\frac{x^2dx}{\sqrt{(a^2-x^2)^3}}=\frac{a^2dx}{\sqrt{(a^2-x^2)^3}} \text{ and } \frac{a^2dx}{\sqrt{(x^2-x^2)^3}}.$$

7. The differential of $\frac{a}{\sqrt{(a^2+x^2)}+x}$ is

$$\frac{-a\left(\frac{xdx}{\sqrt{(a^2+x^2)}}+dx\right)}{[\sqrt{(a^2+x^2)}+x]^2}=-\frac{adx}{a^2+x^2+x\sqrt{(a^2+x^2)}}:$$

noticing, that we shall in like manner get $\frac{adx}{a^2+x^2-x\sqrt{(a^2+x^2)}}$ for the differential of $\frac{a}{\sqrt{(a^2+x^2)}-x}$.

(8.) Supposing X to be a function of x alone, taken for the independent variable; then, since dX, from its definition, equals the sum of all the changes or variations in X, that result from the separate change or variation $x'-x=dx$ of each x in X, it clearly follows that the differential coefficient $\frac{dX}{dx}$ must be independent of dx; and that a double, triple, &c., value of dX must result from a double, triple, &c., value of dx, and so on; and it is clear that the reverse is also true. It is hence evident that we may, according to custom, suppose dx in dX or in dX in the differential coefficient $\frac{dX}{dx}$ to be unlimitedly small, and that when x is the independent variable, dx ought to be regarded as constant or invariable, for otherwise x must be regarded as a function of a variable, and of course it can not be the independent variable.

It is further evident that for $dX = \frac{dX}{dx} dx$, we may, if required, write $dX = \frac{dX}{dx} h$, and regard h as being finite; noticing, that it will generally be very convenient to regard dX in the differential coefficient $\frac{dX}{dx}$ as unlimitedly small, on account of the minuteness of dx.

Calling dX the first differential of X, and $\frac{dX}{dx}$ its *first differential coefficient;* then, if dX or $\frac{dX}{dx}$ contains x, and we take the differential of dX, supposing dx constant, or x to be the independent variable, we shall get $d\,(dX)$, which we shall represent by d^2X, and it will be what is called the second differential of X; and $d\,\frac{dX}{dx} \div dx = \frac{d^2X}{dx^2}$, which is the same as $d^2X \div dx^2 = \frac{d^2X}{dx^2}$, will be what is called the second differential coefficient of X.

In like manner, since $\frac{d^2X}{dx^2}$ is clearly independent of dx, if it contains x, we shall, as before, get $d\,(d^2X) = d^3X$ for the third differential of X, and $\frac{d\,(d^2X)}{dx^2} \div dx = \frac{d^3X}{dx^3}$, which is clearly the same as $d^3X \div dx^3 = \frac{d^3X}{dx^3}$, will clearly be what is called the third differential coefficient of X.

And we may in the same way proceed to find d^4X, $d^5X \ldots d^nX$, which are called the fourth, fifth.... to the nth differential; and the corresponding differential coefficients, $\frac{d^4X}{dx^4}$, $\frac{d^5X}{dx^5} \ldots \frac{d^nX}{dx^n}$. Thus, from $X = x^n$ we get $dX = nx^{n-1}dx$,

$d^2X = n(n-1)x^{n-2}dx^2$, $d^3X = n(n-1)(n-2)x^{n-3}dx^3$, &c., for the first, second, third, &c., differentials; and $\frac{dX}{dx} = nx^{n-1}$, $\frac{d^2X}{dx^2} = n(n-1)x^{n-2}$, $\frac{d^3X}{dx^3} = n(n-1)(n-2)x^{n-3}$, &c., will be the corresponding differential coefficients of x^n.

If we put $x' - x = h$, or $x' = x + h$, and change x in X into x'; then, if the resulting value of X is expressed by X′, it is manifest that X′ is a function of x' or its equal $x + h$.

If X′ is developed into a series arranged according to the ascending powers of h, it is evident (from what has been done) that X and $\frac{dX}{dx}h$ will be the first and second terms of the series, so that we shall have $X' = X + \frac{dX}{dx}h +$, &c.

Since x may stand for any variable, and X for any function of it, it results from the preceding equation, that we can find the first differential of the function by the following

RULE.

Change x in the function into $x + h$, and develop the resulting function into a series arranged according to the ascending powers of h; then the coefficient of h (the simple power of h) in the development, will equal $\frac{dX}{dx}$ the first differential coefficient, which multiplied by dx (supposed unlimitedly small) gives $\frac{dX}{dx}dx$; which is the first differential of the function X.

Thus, if we put $X = x^3$, we get

$$X' = (x + h)^3 = x^3 + 3x^2h + 3xh^2 + h^3;$$

consequently, $3x^2 =$ the first differential coefficient, and of course $dx^3 = 3x^2dx$ is the first differential.

Similarly, from $X = x^{\frac{m}{n}}$ we get

$$X' = (x + h)^{\frac{m}{n}} = x^{\frac{m}{n}} + \frac{m}{n} x^{\frac{m}{n} - 1} h +, \&c.,$$

as is clear from the Binomial Theorem.

Hence, since $\frac{m}{n} x^{\frac{m}{n} - 1}$ is the coefficient of the simple power of h, in the expansion; it follows that we shall have $dx^{\frac{m}{n}} = \frac{m}{n} x^{\frac{m}{n} - 1} dx$ for the differential.

REMARK.—The same differentials of x^3 and $x^{\frac{m}{n}}$, can be obtained immediately from the rule at page 5.

(9.) We will now show how to find the remaining terms of the series, $X' = X + \frac{dX}{dx} h +$, &c.

Thus, by taking the differential of $\frac{dX}{dx} h$, supposing x alone to vary, we have, according to the principles heretofore given, $\frac{d^2X}{dx^2} hh = \frac{d^2X}{dx^2} h^2$, for twice the third term of the series. For any term in $\frac{d^2X}{dx^2} hh$, that results from the multiplication of terms containing h and h taken in any order, will clearly result from the same terms when h and h are interchanged, as is manifest from the manner of obtaining $\frac{d^2X}{dx^2} hh$; consequently, $\frac{d^2X}{dx} \frac{h^2}{1.2}$ is the third term of the series.

Similarly, from $\frac{d^2X}{dx^2} \frac{h^2}{1.2}$, we get $\frac{d^3X}{dx^3} \frac{h^2h}{1.2}$ for thrice the fourth term of the series.

For it is plain that any term in $\frac{d^3X}{dx^3} \frac{h^2h}{1.2}$, that results from the multiplication of a term that contains h^2 by another that contains h, will equally result in two other ways, since h^2 can be formed in two other ways, by combining each h in the first h^2 with the remaining h; consequently, $\frac{d^3X}{dx^3}\frac{h^3}{1.2.3}$ is the fourth term of the series.

It is hence easy to perceive that $\frac{d^4X}{dx^4}\frac{h^4}{1.2.3.4}$ is the fifth term of the series, and so on.

For a more full explanation of the principles used in finding the preceding terms, we shall refer to the solution of Example 16, at p. 56 of my Algebra, and for the common way of finding them, see p. 252 (49.), of the same work: observing, that this method is altogether more complicated than the preceding.

Hence, collecting the terms, we shall have

$$X' = X + \frac{dX}{dx}h + \frac{d^2X}{dx^2}\frac{h^2}{1.2} + \frac{d^3X}{dx^3}\frac{h^3}{1.2.3} +, \&c. \quad . \ . \quad (a);$$

whose law of continuation is manifest: noticing, that h may be positive or negative, according to the nature of the case.

Because X′ is the same function of $x + h$ that X is of x, it follows, if we represent X by $f(x)$ = any function of x, that X′ will become a similar function of $x + h$, represented by $f(x + h)$; consequently, the series (a) becomes

$$f(x+h)=f(x)+\frac{df(x)}{dx}h + \frac{d^2f(x)}{dx^2}\frac{h^2}{1.2}+\frac{d^3f(x)}{dx^3}\frac{h^3}{1.2.3}+, \&c. . (a').$$

(a) and (a') are different forms of what is called *Taylor's Theorem*, which is always true when x and h are undetermined quantities, or when the series does not contain any fractional or negative powers of h. When particular values are assigned to x and h, the series are also true, provided that no term becomes infinite; but if one or more terms become infinite, the series are true no further than to their first infinite terms, exclusively.

If we represent the particular values of X, $\frac{dX}{dx}$, $\frac{d^2X}{dx^2}$, &c., that correspond to $x=0$, by (X), $\left(\frac{dX}{dx}\right)$, $\left(\frac{d^2X}{dx^2}\right)$, &c; then, if we change h into x, and represent the corresponding value of X′ by X, (a) will become

$$X=(X)+\left(\frac{dX}{dx}\right)x+\left(\frac{d^2X}{dx^2}\right)\frac{x^2}{1.2}+\left(\frac{d^3X}{dx^3}\right)\frac{x^3}{1.2.3}+\text{, \&c.} \ldots (b),$$

which is called Maclaurin's Theorem; in which x may be positive or negative, according to the nature of the case. Because X is supposed to be a finite function of x, it clearly follows, if (b) gives an infinite value to any term of X, that (b) is not applicable to the expansion of X.

To perceive the uses of Taylor's and Maclaurin's Theorems, take the following

EXAMPLES.

1. To expand $(x+h)^4$, by Taylor's Theorem.

Here $(x+h)^4$ and x^4 must be used for X′ and X; which give $X=x^4$, $\frac{dX}{dx}=4x^3$, $\frac{d^2X}{dx^2}=12x^2$, $\frac{d^3X}{dx^3}=24x$, $\frac{d^4X}{dx^4}=24$,

and thence (a) becomes

$$(x+h)^4=x^4+4x^3h+6x^2h^2+4xh^3+h^4.$$

2. To expand $(x+h)^n$, according to the ascending powers of h, by Taylor's Theorem.

Here $X'=(x+h)^n$, $X=x^n$, $\frac{dX}{dx}h=nx^{n-1}h$,

$$\frac{d^2X}{dx^2}\frac{h^2}{1.2}=\frac{n(n-1)}{1\,.\,2}x^{n-2}h^2,$$

$$\frac{d^3X}{dx^3}\,\frac{h^3}{1.2.3} = \frac{n(n-1)(n-2)}{1.2.3}x^{n-3}h^3, \text{ \&c.};$$

consequently, from (a) we have $(x+h)^n =$

$$x^n + nx^{n-1}h + \frac{n(n-1)}{1.2}x^{n-2}h^2 + \frac{n(n-1)(n-2)}{1.2.3}x^{n-3}h^3 +, \text{\&c.}$$

3. To expand $X = (a+x)^3$, according to the ascending powers of x, by Maclaurin's Theorem.

Here $X = (a+x)^3$ gives

$$\frac{dX}{dx} = 3(a+x)^2, \quad \frac{d^2X}{dx^2} = 6(a+x), \quad \frac{d^3X}{dx^3} = 6;$$

consequently, putting $x = 0$ in these, we get

$$(X) = a^3, \quad \left(\frac{dX}{dx}\right) = 3a^2, \quad \left(\frac{d^2X}{dx^2}\right) = 6a, \quad \frac{d^3X}{dx^3} = 6;$$

and thence, from (h) we have

$$(a+x)^3 = a^3 + 3a^2x + 3ax^2 + x^3,$$

as required.

4. To expand $\frac{1}{1+x} = (1+x)^{-1}$ according to the ascending powers of x, by Maclaurin's Theorem.

Since $X = (1+x)^{-1}$, we have

$$\frac{dX}{dx} = -(1+x)^{-2}, \quad \frac{d^2X}{dx^2} = 2(1+x)^{-3}, \quad \frac{d^3X}{dx^3} = -2\times 3(1+x)^{-4},$$

and so on; consequently, by putting $x = 0$ in these, we have

$$(X) = 1, \quad \left(\frac{dX}{dx}\right) = -1, \quad \left(\frac{d^2X}{dx^2}\right) = 2, \quad \left(\frac{d^3X}{dx^3}\right) = -2\times 3,$$

and so on. From the substitution of the preceding values in (b), we get

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 +, \text{\&c.},$$

for the required expansion.

5. To expand $\frac{1}{x(1-x)}$, according to the ascending powers of x, by Maclaurin's Theorem.

Because $x=0$ reduces $\frac{1}{x(1-x)}$ to $\frac{1}{0}=$ infinity, it would seem that (*b*) is not applicable to the question. Nevertheless, since (*b*) gives

$$\frac{1}{1-x}=1+x+x^2+x^3+, \&c.,$$

we shall of course have

$$\frac{1}{x(1-x)}=x^{-1}+1+x+x^2+x^3+, \&c.,$$

as required.

6. To expand $X=\frac{a}{x^n}=ax^{-n}$ into a series, arranged according to the ascending powers of x, by Maclaurin's Theorem.

Since $\frac{dX}{dx}=-nax^{-n-1}$, $\frac{d^2X}{dx^2}=n(n+1)x^{-n-2}$, &c., n being positive; and since these are infinite when $x=0$, it is clear that (*b*) is not applicable to the question.

7. To expand X′ by (*a*) on Taylor's Theorem.

Here, by (*a*) we have

$X'=(x+h)^3+(x+h-a)^{-2}$ and $\dot{X}=x^3+(x-a)^{-2}$;

and putting $x=a$, these equations become

$$X'=(a+h)^3+h^{-2}=(a+h)^3+\frac{1}{h^2}, \text{ and } X=a^3+\frac{1}{0^2}.$$

It is hence clear that (*a*) is not applicable to the question any further than to the expansion of $(a+h)^3$.

Remarks.—It is manifest from what has been done, that

$$X'=(a+h)^3+h^{-2}=a^3+h^{-2}+3a^2h+3ah^2+h^3$$

is the true expansion of the proposed expression, when $x=a$. And it is clear that the existence of h^{-2} in the expansion, is the true reason why it can not be found by Taylor's Theorem; since it (the Theorem) evidently supposes the indices of h not only to be integral, but to be positive.

8. To expand $(x+h)^2+[(x+h)^2-a^2]^{\frac{3}{2}}$ according to the ascending powers of h, when $x=a$, by Taylor's Theorem.

Here we have

$$X'=(x+h)^2+[(x+h)^2-a^2]^{\frac{3}{2}},$$

$$X=x^2+(x^2-a^2)^{\frac{3}{2}},$$

$$\frac{dX}{dx}=2x+3x(x^2-a^2)^{\frac{1}{2}},$$

$$\frac{d^2X}{dx^2}=2+3(x^2-a^2)^{\frac{1}{2}}+3x^2(x^2-a^2)^{-\frac{1}{2}}$$

$$=2+3(x^2-a^2)^{\frac{1}{2}}+\frac{3x^2}{(x^2-a^2)^{\frac{1}{2}}}, \text{ \&c.}$$

Because x^2-a^2 with a fractional index enters the denominator of one of the fractional terms of the value of $\frac{d^2X}{dx^2}$, it clearly follows that x^2-a^2 with fractional indices must enter as divisors, in fractional terms of the values of $\frac{d^3X}{dx^3}$, $\frac{d^4X}{dx^4}$, and so on; and it is manifest that like conclusions must hold good in all similar cases.

The substitution of the values of X, $\frac{dX}{dx}$, &c., in (a), gives

$$(x+h)^2+[(x+h)^2-a^2]^{\frac{3}{2}}=x^2+(x^2-a^2)^{\frac{3}{2}}+$$

$$[2x+3x(x^2-a^2)^{\frac{1}{2}}]\,h+\left(2+3(x^2-a^2)^{\frac{1}{2}}+\frac{3x^2}{(x^2-a^2)^{\frac{1}{2}}}\right)\frac{h^2}{1.2}+\text{, \&c.},$$

for the expansion; when no particular value is assigned to x.

When $x = a$, the expansion is easily reduced to

$(x + h)^2 + [(x+h)^2 - a^2]^{\frac{3}{2}} = a^2 + 2ah + h^2 + \frac{3a^2h^2}{1.2\sqrt{0}}$, which is true in its first three terms, but not in its fourth term, which is infinite.

To find the true expansion when $x = a$, we put a for x in $(x + h)^2 + [(x + h)^2 - a^2)^{\frac{3}{2}}$, and thence, by a simple reduction, get $(a + h)^2 + (2ah + h^2)^{\frac{3}{2}}$; which expanded by the Binomial Theorem, or in any other way, gives

$$(a + h)^2 + (2ah + h^2)^{\frac{3}{2}}$$

$$= a^2 + 2ah + (2ah)^{\frac{3}{2}} + h^2 + 3\sqrt{\frac{a}{2}}\, h^{\frac{5}{2}} + \text{, \&c.,}$$

for the true expansion.

REMARKS.—From this and the preceding example, we perceive how the correct expansion may be found when h is affected with a fractional or negative exponent, in consequence of assigning a particular value to x; which are generally said to fall under the failing cases in the applications of Taylor's Theorem: noticing that the Theorem is clearly not applicable in such cases, because it supposes the indices of h in the expansion, to be positive integers.

(10.) If X is a function of any number of variables, as x, y, z, &c.; then, if x, y, &c., are changed successively into $x + h$, $y + i$, $z + k$, &c., the resulting values of X may be expanded into series arranged according to the ascending positive and integral powers of h, i, k, &c., and their products, provided no particular values are assigned to x, y, z, &c.

For let X′ denote the value of X that results from changing x into $x + h$; then (a) p. 16, will be the expansion. In like manner, if y in X, $\frac{dX}{dx}h$, $\frac{d^2X}{dx^2}\frac{h^2}{1.2}$, &c., is changed into

$y + i$, and the resulting values of X, $\frac{dX}{dx} h$, &c., are expanded according to the ascending powers of i, as in (a); then, if X'' stands for the resulting value of X', when y is changed into $y + i$, we shall have

$$\left.\begin{aligned} X'' = X &+ \frac{dX}{dx} h + \frac{d^2X}{dx^2}\frac{h^2}{1.2} + \frac{d^3X}{dx^3}\frac{h^3}{1.2.3} +, \&c. \\ &\frac{dX}{dy} i + \frac{d^2X}{dxdy}\frac{h}{1}\frac{i}{1} + \frac{d^3X}{dx^2dy}\frac{h^2}{1.2}\frac{i}{1} \\ &\frac{d^2X}{dy^2}\frac{i^2}{1.2} + \frac{d^3X}{dxdy^2} h \frac{i^2}{1.2} +, \&c. \\ &+ \frac{d^3X}{dy^3}\frac{i^3}{1.2.3} +, \&c. \end{aligned}\right\} \ldots (a')$$

If we had at first changed y in X into $y + i$, and then expanded as in (a), by arranging the terms according to the ascending powers of i; then, by changing x into $x + h$ in the terms

$$X, \quad \frac{dX}{dy} i, \quad \frac{d^2X}{dy^2} \quad \frac{i^2}{1.2}, \ \&c.,$$

we should in a similar way have obtained the same expansion under another form. Thus, we have

$$\left.\begin{aligned} X'' = X &+ \frac{dX}{dy} i + \frac{d^2X}{dy^2}\frac{i^2}{1.2} + \frac{d^3X}{dy^3}\frac{i^3}{1.2.3} +, \&c. \\ &\frac{dX}{dx} h + \frac{d^2X}{dydx} ih + \frac{d^3X}{dy^2dx}\frac{i^2}{1.2} h +, \&c. \\ &+ \frac{d^2X}{dx^2}\frac{h^2}{1.2} + \frac{d^3X}{dydx^2} i \frac{h^2}{1.2} +, \&c. \\ &+ \frac{d^3X}{dx^3}\frac{h^3}{1.2.3} +, \&c. \end{aligned}\right\} (a'')$$

Because the preceding values of X'' ought clearly to be identical, we must have the equations

$$\frac{d^2X}{dxdy}=\frac{d^2X}{dydx},\ \frac{d^3X}{dx^2dy}=\frac{d^3X}{dydx^2},\ \frac{d^3X}{dxdy^2}=\frac{d^3X}{dy^2dx},$$

$$\ldots\ldots\frac{d^{m+n}X}{dx^m dy^n}=\frac{d^{n+m}X}{dy^n dx^m};$$

which show the differentials indicated in the first and second members of these equations to be identical, as they clearly ought to be, from the nature of the differential calculus.

Changing z into $z+k$ either in (a'') or (a'''), and then proceeding as before, we shall get the expansion that results from changing x, y, z into $a+h$, $y+i$, and $z+k$; and we may proceed in like manner with reference to any number of independent variables that may be contained in X; and the final result will clearly be a generalization of (a) or Taylor's Theorem.

If for h we put dx, (a) becomes

$$X'=X+dX+\frac{d^2X}{1.2}+\frac{d^3X}{1.2.3}+,\ \&c.\ \ldots\ldots\ (a^{\text{IV}});$$

which, according to the preceding generalization of (a), is true when X is a function of any number of independent variables, and that whether the differentials are taken relâtively to all the variables, or not.

EXAMPLES.

1. Given $X=x^3$, to find its differentials:

Here,

$$X=x^3,\ \text{and}\ X'=(x+dx)^3=x^3+3x^2dx+3xdx^2+dx^3;$$

consequently, since

$$X'=X+dX+\frac{d^2X}{1.2}+,\ \&c.,$$

we have

$$X + dX + \frac{d^2X}{1.2} + \frac{d^3X}{1.2.3} = x^3 + 3x^2dx + 3xdx^2 + dx^3,$$

which must clearly be an identical equation.

Hence, from a comparison of like terms, we get

$$X = x^3,\ dX = 3x^2dx,\ d^2X = 1.2.3\,xdx = 6xdx,\ d^3X = 6dx^3,$$

as required.

2. To find the differentials of $X = x^{\frac{1}{2}}$.

Since $$X' = X + dX + \frac{d^2X}{1.2} +, \&c.,$$

$$= (x + dx)^{\frac{1}{2}} = x^{\frac{1}{2}} + \frac{1}{2}\frac{dx}{x^{\frac{1}{2}}} - \frac{1}{8}\frac{dx^2}{x^{\frac{3}{2}}} + \frac{1}{16}\frac{dx^3}{x^{\frac{5}{2}}} -, \&c.;$$

consequently, from a comparison of like terms, we have

$$X = x^{\frac{1}{2}},\ dX = \frac{dx}{2x^{\frac{1}{2}}},\ d^2X = -\frac{dx^2}{4x^{\frac{3}{2}}},\ d^3X = \frac{3dx^3}{8x^{\frac{5}{2}}},$$

and so on, indefinitely.

3. To find the differentials of $X = xy$.

Since $$X' = X + dX + \frac{d^2X}{1.2} = (x + dx).(y + dy)$$

$$= xy + ydx + xdy + dxdy,$$

an identical equation: its like terms equated, give

$$X = xy,\ dX = ydx + xdy,\ \text{and}\ d^2X = 2dxdy,$$

as required.

4. To find the differentials of $X = xy^2$.

From $$X' = X + dX + \frac{d^2X}{1.2} + \frac{d^3X}{1.2.3} = (x + dx).(y + dy)^2$$

$$= xy^2 + 2xydy + y^2dx + xdy^2 + 2ydxdy + dxdy^2,$$

we readily get

$$X = xy^2, \qquad dX = 2xydy + y^2dx,$$

$$d^2X = 2xdy^2 + 4ydxdy, \qquad d^3X = 6dxdy^2.$$

Again, if we put x and y in (a'') each equal to naught, and represent the corresponding values of

$$X'', \ X, \ \frac{dX}{dx}, \ \frac{dX}{dy}, \ \&c., \text{ by } X, \ (X), \ \left(\frac{dX}{dx}\right), \ \left(\frac{dX}{dy}\right), \ \&c.;$$

then changing h and i into x and y, (a'') becomes

$$\left.\begin{aligned} X = (X) + \left(\frac{dX}{dx}\right) x + \left(\frac{d^2X}{dx^2}\right)\frac{x^2}{1.2} +, \&c. \\ + \left(\frac{dX}{dy}\right) y + \left(\frac{d^2X}{dxdy}\right) xy +, \&c. \\ + \left(\frac{d^2X}{dy^2}\right) \frac{y^2}{1.2} +, \&c. \end{aligned}\right\} \quad \ldots . (b');$$

which is Maclaurin's Theorem extended to the expansion of a function of two independent variables: and it is easy to perceive how Maclaurin's Theorem may be extended to the expansion of a function of any number of independent variables.

To illustrate (b'), we will apply it to the expansion of $X = (ax + by)^2$.

Here $\quad \dfrac{dX}{dx} = 2a\,(ax + by), \ \dfrac{d^2X}{dx^2} = 2a^2, \ \dfrac{d^3X}{dx^3} = 0, \ \&c.;$

$$\frac{dX}{dy} = 2b\,(ax + by), \ \frac{d^2X}{dy^2} = 2b^2, \ \frac{d^3X}{dy^3} = 0, \ \&c.;$$

and $\quad \dfrac{d^2X}{dxdy} = 2ab, \ \dfrac{d^2X}{dydx} = 2ab.$

Putting $x = 0$ and $y = 0$ in X and these values, we get

$$(X) = 0, \ \left(\frac{dX}{dx}\right) = 0, \ \left(\frac{d^2X}{dx^2}\right) = 2a^2, \ \frac{d^3X}{dx^3} = 0, \&c.;$$

also, $\quad \left(\dfrac{dX}{dy}\right) = 0, \ \left(\dfrac{d^2X}{dy^2}\right) = 2b^2, \ \dfrac{d^3X}{dy^3} = 0, \ \&c.;$

$$\left(\frac{d^2X}{dxdy}\right) = 2ab.$$

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Substituting these values in (b'), we get

$$X = (ax + by)^2 = \frac{2a^2x^2}{1.2} + 2abxy + \frac{2b^2y^2}{1.2} = a^2x^2 + 2abxy + b^2y^2;$$

which is evidently correct.

REMARKS.—1. If $ax + by$ is eliminated from the equations

$$\frac{dX}{dx} = 2a\,(ax + by) \text{ and } \frac{dX}{dy} = 2b\,(ax + by),$$

we get
$$a\frac{dX}{dy} - b\frac{dX}{dx} = 0,$$

which is called an equation of partial differences; but, since $\frac{dX}{dy}$ and $\frac{dX}{dx}$ are differential coefficients, it is clearly more correct to call it an equation of partial differential coefficients.

2. If $X = f(ax + by) =$ some function of $ax + by$, it is easy to perceive that $\frac{dX}{dx}$ and $\frac{dX}{dy}$ will be of the forms

$$\frac{dX}{dx} = \frac{df(ax + by)}{dx} = af'(ax + by)$$

and
$$\frac{dX}{dy} = \frac{df(ax + by)}{dy} = bf'(ax + by).$$

It is easy to perceive that the elimination of $f'(ax + by)$ from these equations, gives the equation $a\frac{dX}{dy} - b\frac{dX}{dx} = 0$, the same as in 1, so that it does not depend on the form of f. Reciprocally, if in any calculation we meet with an equation of the form $a\frac{dX}{dy} - b\frac{dX}{dx} = 0$, it may clearly be supposed to have been derived from an equation of the form $X = f(ax + by) =$ an arbitrary function of $ax + by$.

3. If $a = b$, the preceding equation gives $\frac{dX}{dy} = \frac{dX}{dx}$; consequcntly, if $X = f(x+y)$ = a function of $x + y$, *it follows that the partial differential coefficients, when taken separately with reference to x and y, must equal each other.* For more ample details relatively to the preceding remarks, &c., we shall refer to Art. 77, &c., p. 230, vol. 1, of the "Calcul Différentiel et Intégral," of Lacroix.

(11.) If X and Y represent functions of any number of independent variables, whether the variables in X and Y are the same or not; then, we propose to show how to find any differential of the product XY.

Thus, by indicating and taking the differentials of XY in succession, we get

$$d\,(XY) = XdY + YdX,$$

$$d^2\,(XY) = Xd^2Y + 2dXdY + Yd^2X,$$

$$d^3\,(XY) = Xd^3Y + 3dXd^2Y + 3d^2XdY + d^3XY, \ \ldots \text{ to}$$

$$d^n\,(XY) = Xd^nY + ndXd^{n-1}Y + \frac{n(n-1)}{1.2}\,d^2Xd^{n-2}Y + \frac{n(n-1)(n-2)}{1.2.3}\,d^3Xd^{n-3}Y +, \&c. \ldots (c);$$

where it is clear that n denotes an integer, such that the nth differential of the product, indicated by $d^n(XY)$ in the first member of the equation, is developed in the second member, and of course the equation must be considered as being an identical equation.

It is clear that (c) can be obtained immediately, from the development of $(dY + dX)^n$ according to the descending powers of dY and the ascending powers of dX, by the Binomial Theorem; being particular, in the development, to apply the exponents of the powers of dY and dX to the characteristic d, and to write Y for d^0Y, and X for d^0X. (See Art. 91, p. 256, Vol. 1, of the "Calcul Différentiel," &c., of Lacroix.)

REMARKS.—1. It is clear that the differentials of the quotient $\frac{X}{Y} = XY^{-1}$ may be found in much the same way as before, by changing Y into Y^{-1}.

2. If dX, dY, dZ, &c., stand for the differentials of any number of functions, X, Y, Z, &c.; then the differential of the product indicated by d^n(XYZ, &c.), will be obtained from the power $(dX + dY + dZ +, \text{\&c.})^n$, in a way similar to that of obtaining the differential indicated by d^n(YX) from $(dY + dX)^n$, as explained above. For further information on what has been done, see Lacroix and "Théorie Analytique des Probabilités" of Laplace.

To illustrate what has been done, take the following

EXAMPLES.

1. To find the differential of XY, indicated by d^3(XY).

Here, by putting 3 for n in (c), we immediately get

$$d^3(XY) = Xd^3Y + 3dXd^2Y + 3d^2XdY + d^3XY:$$

which can also be found from

$$(dY + dX)^3 = (dY)^3 + 3(dY)^2dX + 3dY(dX)^2 + (dX)^3,$$

as has been stated; noticing that for $(dY)^3 = 1 \times (dY)^3$, we ought to write [since $(dX)^0 = 1$] $(dX)^0(dY)^3$, and for $(dX)^3$, we must also write $(dX)^3(dY)^0$.

For, by changing $(dX)^0(dY)^3$ into Xd^3Y, and $3(dY)^2dX$ into $3d^2YdX$, and so on, we shall, as before, get

$$d^3(XY) = Xd^3Y + 3dXd^2Y + 3d^2XdY + d^3XY.$$

2. To develop $d^3(x^3y^2)$, by means of the preceding formula.

By the substitution of x^3 for X, and y^2 for Y, it immediately changes into

$$d^3(x^3y^2) = 18x^2dxdy^2 + 36xy(dx)^2dy + 6y^2(dx)^3,$$

since $x^3d(dy)^2 = 0$, on account of the supposed constancy of dy, y being regarded as an independent variable.

3. To find the second differential of $\frac{X}{Y}$, or to expand $d^2(XY^{-1})$, when $X = x^2$ and $Y = y$.

Here, since $d^2(XY^{-1}) = Xd^2(Y^{-1}) + 2dXd(Y^{-1}) + d^2XY^{-1}$, by putting x^2 for X and y for Y, and performing the indicated differentiation, we get

$$d^2(x^2y^{-1}) = 2x^2y^{-3}dy^2 - 4xy^{-2}dxdy + 2y^{-1}dx^2$$
$$= \frac{2x^2dy^2}{y^3} - \frac{4xdxdy}{y^2} + \frac{2dx^2}{y}.$$

(12.) If $y = f(z)$ and $z = f'(x)$, we now propose to show how to find the differential coefficient of y regarded as a function of x.

Here we clearly have $dy = \frac{df(z)}{dz}dz$, and $dz = \frac{df'x}{dx}dx$, and consequently, by substituting the value of dz from the second in the first, we have $dy = \frac{df(z)}{dz} \times \frac{df'(x)}{dx} \times dx$; which gives $\frac{dy}{dx} = \frac{df(z)}{dz} \times \frac{df'(x)}{dx}$; as required.

It is easy to perceive that if $y = f(z)$, $z = \phi(v)$, $v = \psi(x)$, we shall in like manner get

$$dy = \frac{df(z)}{dz} \times \frac{d\phi(v)}{dv} \times \frac{d\psi(x)}{dx}dx,$$

or $$\frac{dy}{dx} = \frac{df(z)}{dz} \times \frac{d\phi(v)}{dv} \times \frac{d\psi(x)}{dx} \quad \ldots\ldots\ldots\ldots (d);$$

and so on, to any extent.

EXAMPLES.

1. Given $y = 3z^2$, $z = 4x^3$, to find the differential of y or its differential coefficient, regarding it as a function of x,

Here, from $dy = 6zdz$ and $dz = 12x^2dx$, we get by substitution, $dy = 72zx^2dx$, or $\frac{dy}{dx} = 72zx^2$.

2. Given $y = az^2$, $z = bv^3$, and $v = cx^4$, to find dy, or its differential coefficient regarded as a function of x.

Here, we have $dy = 2azdz$, $dz = 3bv^2dv$, and $dv = 4cx^3dx$; consequently, by substitution, as in (d), we shall have

$$dy = 24abczv^2x^3dx, \text{ or } \frac{dy}{dx} = 24abczv^2x^3.$$

3. To simplify the differential of $y = (ax^3 + bx^2 + c)^4$ or its differential coefficient, by putting $z = ax^3 + bx^2 + c$, which reduces the proposed equation to $y = z^4$.

Here, from $y = z^4$ we have $dy = 4z^3dz$, and from $z = ax^3 + bx^2 + c$ we have $dz = 3ax^2dx + 2bxdx$; consequently, from the substitution of dz, we have

$$dy = 4z^3 \times (3ax^2dx + 2bxdx),$$

or
$$\frac{dy}{dx} = 4(ax^3 + bx^2 + c)^3 \times (3ax^2 + 2bx);$$

which is the same result that the immediate differentiation of the proposed equation will give.

REMARKS.—1. Thus we perceive how we may often simplify the differentials or differential coefficients of complicated expressions.

2. If we have $y = f(u.z)$, such that we have $u = \phi(x)$ and $z = \psi(x)$; then, we shall clearly have

$$dy = \frac{df(u.z)}{du}du + \frac{df(u.z)}{dz}dz,$$

and
$$du = \frac{d\phi(x)}{dx}dx, \quad dz = \frac{d\psi(x)}{dx}dx,$$

which are clearly the same as

$$dy = \frac{dy}{du}\,du + \frac{dy}{dz}\,dz, \text{ and } du = \frac{du}{dx}\,dx, \quad dz = \frac{dz}{dx}\,dx.$$

Hence, eliminating du and dz from dy, it will become

$$dy = \frac{dy}{du} \cdot \frac{du}{dx}\,dx + \frac{dy}{dz} \cdot \frac{dz}{dx}\,dx,$$

or

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} + \frac{dy}{dz} \cdot \frac{dz}{dx}.$$

In much the same way, if we have

$$y = f(t, v, z, \&c.), \ t = \mathrm{F}(x), \ v = \phi(x), \ z = \psi(x), \&c.;$$

then, as before, we shall clearly have

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} + \frac{dy}{dv} \cdot \frac{dv}{dx} + \frac{dy}{dz} \cdot \frac{dz}{dx} +, \&c. \ldots (e).$$

It is easy to perceive that we may use (e) to simplify the differentiation of complicated functions, in a way very analogous to that of (d).

Thus, to find the differential or differential coefficients of

$$y = [\sqrt{(a^2 + x^2)} - \sqrt{(a^2 - x^2)}]^{\frac{1}{2}},$$

we put $u = \sqrt{(a^2 + x^2)}$, $z = \sqrt{(a^2 - x^2)}$, and thence get $y = \sqrt{(u - z)}$. And

$$dy = \frac{du - dz}{2\sqrt{(u - z)}}, \quad du = \frac{xdx}{\sqrt{(a^2 + x_2)}}, \text{ and } dz = -\frac{xdx}{\sqrt{(a^2 - x^2)}};$$

consequently, substituting the values of du and dz, and restoring the values of u and z in $\sqrt{(u - z)}$, we have

$$dy = \frac{1}{2}\left(\frac{xdx}{\sqrt{(a^2 + x^2)}} + \frac{xdx}{\sqrt{(a^2 - x^2)}}\right) [\sqrt{(a^2 + x^2)} - \sqrt{(d^2 + x^2)}]^{-\frac{1}{2}},$$

or

$$\frac{dy}{dx}\,dx = \frac{\dfrac{xdx}{\sqrt{(a^2 + x^2)}} + \dfrac{xdx}{\sqrt{(a^2 - x^2)}}}{2[\sqrt{(a^2 + x^2)} - \sqrt{(a^2 - x^2)}]^{\frac{1}{2}}},$$

the same that the immediate differentiation of the proposed equation will give.

(13.) Supposing two or more variables to be connected by any equation, or that each of the variables is (in virtue of the equation) an implicit function of all the rest; then, it is proposed to show how to find the differential equation that exists among the variables; and the differential coefficients that result from considering either of the variables as being a function of each of the others.

1. Let $X = f(x, y) = 0$, stand for any equation between x and y; then, since X may clearly be treated as being an explicit function of x and y considered as being independent variables, we shall have

$$dX = \frac{dX}{dx}\,dx + \frac{dX}{dy}\,dy;$$

or, since $dX = 0$, the equation is reduced to

$$\frac{dX}{dx}\,dx + \frac{dX}{dy}\,dy = 0.$$

From this equation we have

$$\frac{dx}{dy} = -\frac{\frac{dX}{dy}}{\frac{dX}{dx}};$$

in which x is regarded as a function of y: and, by taking the reciprocal of this equation, we have

$$\frac{dy}{dx} = -\frac{\frac{dX}{dx}}{\frac{dX}{dy}} \quad \ldots\ldots\ldots\ldots (f);$$

in which y is regarded as being a function of x.

Thus, if $X = y - 3x$ we have

$$\frac{dX}{dy} = \frac{dy}{dy} = 1, \text{ and } \frac{dX}{dx} = -3,$$

which reduce the first of the preceding equations to $\frac{dX}{dy} = \frac{1}{3}$ and the second to $\frac{dy}{dx} = 3$. It is easy to perceive that the equation $y - 3x = 0$ or $y = 3x$, immediately gives

$$\frac{dx}{dy} = \frac{1}{3} \text{ or } \frac{dy}{dx} = 3,$$

the same as before, and found in a much more simple manner.

Similarly, from $X = x^2 + y^2 - r^2 = 0$ we get

$$\frac{dX}{dy} = 2y \text{ and } \frac{dX}{dx} = 2x;$$

consequently, we have

$$\frac{dx}{dy} = -\frac{\frac{dX}{dy}}{\frac{dX}{dx}} = -\frac{y}{x} \quad \text{or} \quad \frac{dy}{dx} = -\frac{x}{y}.$$

Again, by differentiating the equation $x^2 + y^2 - r^2 = 0$, we have $2xdx + 2ydy = 0$; which gives $\frac{dx}{dy} = -\frac{y}{x}$ or $\frac{dy}{dx} = -\frac{x}{y}$, the same as before.

2. Supposing $X = 0$ to be a function of any number of variables, x, y, z, &c., then (as before) we shall have the differential equation

$$\frac{dX}{dx}dx + \frac{dX}{dy}dy + \frac{dX}{dz}dz +, \text{\&c.}, = 0.$$

Supposing z to be regarded as being a function of each of the other variables, then the partial differential equation between x and z, gives $\frac{dX}{dx}\,dx + \frac{dX}{dz}\,dz = 0$,

which gives $$\frac{dz}{dx} = -\frac{\frac{dX}{dx}}{\frac{dX}{dz}};$$

and in like manner, $$\frac{dz}{dy} = -\frac{\frac{dX}{dy}}{\frac{dX}{dz}},$$ and so on.

REMARK.—*The elimination of a constant from an equation by means of its differential equation, generally changes the form of the differential coefficient.*

Thus, by taking the differential of $y^2 = ax$, we get

$$2ydy = adx,$$

which gives $$a = \frac{2ydy}{dx};$$

consequently, substituting this value for a in the proposed equation, we have

$$2xdy = ydx \quad \text{or} \quad \frac{dy}{dx} = \frac{y}{2x}.$$

Hence the differential coefficients

$$\frac{dy}{dx} = \frac{a}{2y} \quad \text{and} \quad \frac{dy}{dx} = \frac{y}{2x},$$

although equivalent, are of different forms.

(14.) *When a variable is a function of any variables regarded as independent; then, in taking the second, third, &c., differentials of the function, the differentials of its independent variables must each be constant or invariable. What is here said, is clear from what is shown at pages*

11 *and* 12, *and clearly results from the nature of the case.*

Hence, we deduce the following conclusions:

1. *The* nth *differential of an explicit function of any number of independent variables, is either equal to the sum of terms, that contain* n *dimensions of the differentials of the independent variables, or it is equal to naught.*

Thus, if $y = \frac{x^2}{a}$, and x is the independent variable, we have $dy = \frac{2xdx}{a}$, $d^2y = \frac{2(dx)^2}{a}$, $d^3y = 0$, $d^4y = 0$; all the differentials above the second being equal to naught, since x being the independent variable dx is constant or invariable.

2. *The* nth *differential of an implicit function of any number of independent variables mixed together, must be such that there shall be* n *differentials in each of its terms.*

Thus, from $yx = a^2$, regarding y and x as functions of other variables; we have

$$ydx + xdy = 0,$$

$$xd^2y + 2dxdy + yd^2x = 0,$$

$$xd^3y + 3dxd^2y + 3d^2xdy + d^3xy = 0,$$

and so on, as at page 27.

If we proceed in like manner with the equation

$$y^2 + x^2 - r^2 = 0,$$

we get

$$2ydy + 2xdx = 0$$

(for which we may put $ydy + xdx = 0$),

$$yd^2y + dy^2 + xd^2x + dx^2 = 0,$$

$$yd^3y + 3dyd^2y + 3dxd^2x + xd^3x = 0,$$

and so on.

If x is taken for the independent variable, or if dx is constant or invariable, these equations will be reduced to the more simple forms

$$ydy + xdx = 0,$$
$$yd^2y + dy^2 + dx^2 = 0,$$

and

$$yd^3y + 3dyd^2y = 0.$$

In like manner if x is regarded as a function of y, we have

$$ydy + xdx = 0,$$
$$xd^2x + dx^2 + dy^2 = 0,$$
$$xd^3x + 3dxd^2x = 0,$$

and so on.

Again, if x and y in the product xy^2, are considered and treated as being independent variables, then we shall have

$$d\,(xy^2) = y^2dx + 2xydy,$$
$$d^2(xy^2) = 4ydydx + 2xdy^2$$
$$d^3(xy^2) = 6dxdy^2,$$
$$d^4(xy^2) = 0,$$
$$d^5(xy^2) = 0,$$

and so on.

(15.) *Having an equation between x and y, in which x is the independent variable; we propose to show how to change the equation, so that x shall become a function of y, or so that x and y shall become functions of a new variable.*

We may, according to what is shown at pages 10 and 12, represent that y is a function of x by the form $\dfrac{dy}{dx} = \dfrac{1}{\dfrac{dx}{dy}}$; consequently, we may differentiate the first member of this by regarding x as being the independent variable, and the second member on the supposition that x is a function of y; which gives $\dfrac{d^2y}{dx} = -\dfrac{d^2xdy}{dx^2}$. It is easy to perceive that this result is the same as to write $d\dfrac{dy}{dx}$ for $\dfrac{d^2y}{dx}$, and then to differentiate $d\,\frac{dy}{dx}$ by regarding dy as constant, or taking y for the independent variable.

Indeed, if we had differentiated the right number of the above equation by regarding dx and dy as both variable, we should have found $d\left\{\frac{1}{\frac{dx}{dy}}\right\} = \frac{dxd^2y - dyd^2x}{dx^2}$, which is clearly the same as to take the differential of $\frac{dy}{dx}$, when dy and dx are both regarded as variable; consequently if we make $d^2y = 0$, or $dy =$ const., the preceding differential reduces to $-\frac{d^2xdy}{dx^2}$, as found above.

It is hence manifest that for $\frac{d^2y}{dx}$ we may write $d\,\frac{dy}{dx}$, or regard dy and dx as both variable, or if convenient take $-\frac{d^2xdy}{dx^2}$; and *vice versa.*

(16.) *To show the facility that the use of partial differential coefficients sometimes gives in the solution of difficult questions, we will take the following important*

PROBLEM.

Given $z = \phi\,(a + xy) =$ a function of $a + xy$ (1), in which a and x are regarded as independent variables, and $y = \psi\,(z) =$ a function of z (2); then it is proposed to develop z according to the ascending powers of x.

According to Maclaurin's Theorem, see (b) page 17, we may assume

$$z = z' + \frac{dz'}{dx}\,x + \frac{d^2z'}{dx^2}\,\frac{x^2}{1.2} + \frac{d^3z'}{dx^3}\,\frac{x^3}{1.2.3} +, \&c. \ldots (3),$$

for the development, in which z', $\frac{dz'}{dx}$, $\frac{d^2z'}{dx^2}$, &c., represent the values of z, $\frac{dz}{dx}$, $\frac{d^2z}{dx^2}$, &c., that result from putting $x=0$ in them.

It is manifest by taking the partial differential coefficients of (1), that we shall have

$$\frac{dz}{dx}=\frac{d.\phi(a+xy)}{d.(a+xy)}\times\frac{d.(a+xy)}{dx},$$

and $$\frac{dz}{da}=\frac{d.\phi(a+xy)}{d.(a+xy)}\times\frac{d.(a+xy)}{da};$$

consequently, eliminating the differential coefficient

$$\frac{d.\phi(a+xy)}{d.(a+xy)}$$

from these, we shall have

$$\frac{dz}{dx}\times\frac{d.(a+xy)}{da}=\frac{dz}{da}\times\frac{d.(a+xy)}{dx},$$

or $$\frac{dz}{dx}\left(1\times x\frac{da}{dy}\right)-\frac{dz}{da}\left(y\times x\frac{dy}{dx}\right).$$

Because (2) gives

$$\frac{dy}{da}=\frac{d\psi(z)}{dz}\,\frac{dz}{da}=\frac{dy}{dz}\cdot\frac{dz}{da},\quad\text{and}\quad\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx},$$

we easily reduce the preceding equation to

$$\frac{dz}{dx}=y\frac{dz}{da}\,.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;(4).$$

By taking the partial differential coefficients of (4) relatively to x, we shall have

$$\frac{d^2z}{dx^2}=d\left(y\frac{dz}{da}\right)\div dx=\frac{dy}{dx}\cdot\frac{dz}{da}+yd\left(\frac{dz}{da}\right)\div dx;$$

or, since

$$\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}\quad\text{and}\quad yd\left(\frac{dz}{da}\right)\div dx=y\frac{d^2z}{dadx}=y\frac{d^2z}{dxda}$$

(page 22), we have

$$\frac{d^2z}{dx^2} = \frac{dy}{dz} \cdot \frac{dz}{da} \cdot \frac{dz}{dx} + y \frac{d^2z}{dxda}$$

$$= \left(\text{since } \frac{dy}{dz} \cdot \frac{dz}{da} = \frac{dy}{da}\right) \; d\left(y \frac{dz}{dx}\right) \div da;$$

and since (4) gives $\frac{dz}{dx} = y \frac{dz}{da}$, this is easily reduced to

$$\frac{d^2z}{dx^2} = d\left(y^2 \frac{dz}{da}\right) \div da = \frac{d\left(y^2 \frac{dz}{da}\right)}{da}$$

Differentiating the members of this equation relatively to x, we have

$$\frac{d^3z}{dx^3} = \frac{d^2\left(y^2 \frac{dz}{da}\right)}{dadx} = \frac{d^2\left(y^2 \frac{dz}{da}\right)}{dxda} = d\left\{\frac{d\left(y^2 \frac{dz}{da}\right)}{dx}\right\} \div da,$$

on account of the independence of a and x, and the differentiations relatively to them.

It is easy to perceive that $\frac{d\left(y^2 \frac{dz}{da}\right)}{dx}$ may, as before, be reduced to $d\left(y^3 \frac{dz}{da}\right) \div da$, which gives

$$\frac{d^3z}{dx^3} = d^2\left(y^3 \frac{dz}{da}\right) \div da^2;$$

and proceeding with this, as before, we have

$$\frac{d^4z}{dx^4} = d^2\left\{\frac{d\left(y^3 \frac{dz}{da}\right)}{dx}\right\} \div da^2,$$

which, as before, gives

$$\frac{d^4z}{dx^4} = d^3\left(y^4 \frac{dz}{da}\right) \div da^3, \text{ and so on.}$$

If the values of $\frac{dz}{da}$ and y, that result from putting $x = 0$ in them, are represented by $\frac{dz'}{da}$ and y', we shall have

$$\frac{dz'}{dx} = y'\frac{dz'}{da},\ \frac{d^2z'}{dx^2} = \frac{d\left(y'^2\frac{dz'}{da}\right)}{da},\ \text{and so on.}$$

Hence, from the substitution of these values in (3), we get

$$z = z' + \left(y'\frac{dz'}{da}\right)x + \frac{d\left(y'^2\frac{dz'}{da}\right)}{da}\frac{x^2}{1.2} + \frac{d^2\left(y'^3\frac{dz'}{da}\right)}{da^2}\frac{x^3}{1.2.3} +, \&c. \quad \ldots\ldots\ldots (h);$$

which clearly holds good, when any like functions of z and z' are put for z and z' in it; noticing, that (h), thus generalized, is called the *Theorem of Laplace;* and if we put 1 for x, in (h), it will become what is called the *Theorem of La Grange.*

To perceive some of the uses of (h), take the following

EXAMPLES.

1. Given $bz^n - cz + d = 0$, to find z in a series of the known quantities.

The equation is readily changed to the form

$$z = \frac{d}{c} + \frac{b}{c}z^n = a + xy;$$

consequently, for ϕ in (1) we put 1, or unity, and $y = z^n$;

also, $\quad a = \frac{d}{c} \quad \text{and} \quad x = \frac{b}{c}.$

By putting $x = 0$ we get $z' = a$ and thence $\frac{dz'}{da} = 1$;

also, $\quad y = z'^n \quad \text{gives} \quad y' = z'^n = a^n,$

and thence $\quad y'\frac{dz'}{da} = a^n.$

Because $y' = a^n$ and $\frac{dz'}{da} = 1$, $d\left(y'^2 \frac{dz'}{da}\right) \div da$

becomes $$\frac{d\,(a^{2n})}{da} = 2na^{2n-1};$$

also, $d^2\left(y'^3 \frac{dz'}{da}\right) \div da^2 = d^2\,(a^{3n}) \div da^2 = 3n\,(3n - 1)\,a^{3n-2}$,

and so on.

Hence, collecting the results, we shall get

$$z = a + a^n x + 2na^{2n-1}\,\frac{x^2}{1.2} + 3n(3n-1)\,a^{3n-2}\,\frac{x^3}{1.2.3} +, \text{\&c.}$$

If $\quad n = 3,\ b = 1,\ c = 3,\ \text{ and }\ d = -1,$

the proposed equation becomes $z^3 - 3z - 1 = 0$;

which gives $a = -\frac{1}{3}$, and $x = \frac{1}{3}$.

Hence, from the preceding series, we shall have

$$z = -\frac{1}{3} - \frac{1}{81} - \frac{1}{729} - \frac{4}{19683} -, \text{\&c.}$$

$$= -\frac{6835}{19683} -, \text{\&c.} = -0.3472,$$

which is one of the roots of the equation $z^3 - 3z - 1 = 0$; correctly found in all its figures.

2. Given $bz - cz^{\frac{1}{n}} + d = 0$, to develop z in a series.

Since the equation is equivalent to

$$z = -\frac{d}{b} + \frac{c}{b}\ z^{\frac{1}{n}} = a + xy;$$

we have $\phi = 1$, $a = -\frac{d}{b}$, $x = \frac{c}{b}$, $y = z^{\frac{1}{n}}$. Putting $x = 0$, get $z' = a$ and $\frac{dz'}{da} = 1$; and $y = z^{\frac{1}{n}}$ becomes $y' = z'^{\frac{1}{n}} = a^{\frac{1}{n}}$.

Hence, if we change n into $\frac{1}{n}$ in the series for z in the preceding example, we shall get

$$z = a + a^{\frac{1}{n}} x + \frac{2}{n} a^{\frac{2}{n}-1} \frac{x^2}{1.2} + \frac{3}{n}\left(\frac{3}{n} - 1\right) a^{\frac{3}{n}-2} \frac{x^3}{1.2.3}, +, \&c.,$$

as required.

If we have the equation $3v^3 - v - 1 = 0$; then, putting

$$v^3 = z \quad \text{or} \quad v = z^{\frac{1}{3}},$$

it becomes $3z - z^{\frac{1}{3}} - 1 = 0$ or $z = \frac{1}{3} + \frac{1}{3} z^{\frac{1}{3}}$,

so that $n = 3$, $a = \frac{1}{3}$ and $x = \frac{1}{3}$ in this equation.

If $\frac{1}{3}$ is put for $\frac{1}{n}$ a and x in the series for z, it becomes

$$z = \frac{1}{3} + \left(\frac{1}{3}\right)^{\frac{4}{3}} + \left(\frac{1}{3}\right)^{\frac{8}{3}} +, \&c.$$

$= 0.3333 + 0.23112 + 0.053416 +, \&c. = 0.61787 +, \&c.$; and hence $v = \sqrt[3]{z} = \sqrt[3]{0.61787} = 0.85173$, whose first two decimal places are correct.

3. Given $Az^n + Bz^{n'} + Cz^{n''} + \ldots . + N = 0$, to find z.

Since the equation is equivalent to

$$z^n = -\frac{N}{A} - \frac{1}{A}(Bz^{n'} + Cz^{n''} +, \&c.) = a + xy;$$

we have $a = -\frac{N}{A}$ and $xy = -\frac{1}{A}(Bz^{n'} + Cz^{n''} +, \&c.)$; and we may evidently put $x = -\frac{1}{A}$ and $y = Bz^{n'} + Cz^{n''} +, \&c.$

From what precedes, we get $z = (a + xy)^{\frac{1}{n}}$, which corresponds to $\phi(a + xy)$ in (1), p. 37; which, by putting $x = 0$, gives $z' = a^{\frac{1}{n}}$, which gives

$$\frac{dz'}{da} = \frac{1}{n} a^{\frac{1}{n}-1} = \frac{a^{\frac{1-n}{n}}}{n},$$

and $\quad y' = \mathrm{B}z'^{n'} + \mathrm{C}z'^{n''} +, \&c. = \mathrm{B}a^{\frac{n'}{n}} + \mathrm{C}a^{\frac{n''}{n}} +, \&c.$

Hence, from (h) we get

$$z = a^{\frac{1}{n}} + (\mathrm{B}a^{\frac{n'}{n}} + \mathrm{C}a^{\frac{n''}{n}} +, \&c.)\, \frac{a^{\frac{1-n}{n}}}{n}\, x +$$

$$\frac{d\left[(\mathrm{B}a^{\frac{n'}{n}} + \mathrm{C}a^{\frac{n''}{n}} +, \&c.)^2\, a^{\frac{1-n}{n}}\right]}{da}\, \frac{x^2}{1.2n} +$$

$$\frac{d^2\left[(\mathrm{B}a^{\frac{n'}{n}} + \mathrm{C}a^{\frac{n''}{n}} +, \&c.)^3 a^{\frac{1-n}{n}}\right]}{da^3}\, \frac{x^2}{1.2.3\,n} +, \&c.$$

To illustrate this formula, we shall take the equation

$$z^3 - 3z - 1 = 0, \text{ under the form } z^2 - z^{-1} - 3 = 0.$$

Hence, $\quad \mathrm{A} = 1,\ \mathrm{B} = -1,\ \mathrm{C} = 0,\ \mathrm{D} = 0,\ \&c.,$

$$\mathrm{N} = -3,\ a = -\frac{\mathrm{N}}{\mathrm{A}} = 3,\ x = -1,\ n = 2,\ n' = -1,\ n'' = 0, \&c.$$

From the formula, we get

$$z = \sqrt{a} + \frac{1}{na} - \frac{n+1}{2n^2 a^{\frac{2n+2}{n}}} +, \&c.;$$

consequently, by putting 3 for a and 2 for n, and giving the square roots the ambiguous sign $\pm$, we get

$$z = \pm\sqrt{3} + \frac{1}{6} \pm \frac{1}{72} +, \&c.$$

$$= \pm 1.7320 + 0.1666 \mp 0.0138 +, \&c.$$

Hence, we have $1.88 +$ and $-1.54 -$ for approximate values of two of the roots of the proposed equation, correctly found to two places of figures in each.

REMARKS.—1. It is sometimes necessary to distinguish between *total* and *partial* differential coefficients. Thus, if we have

$$\frac{du}{dx}=\frac{du}{dp}\frac{dp}{dx}+\frac{du}{dq}\frac{dq}{dx}+\frac{du}{dr}\frac{dr}{dx},$$

we call the first member of the equation the *total differential coefficient*, and the terms that compose its right member are its parts, or what are called the partial differential coefficients.

2. If $p=x$, it is clear that the equation will be reduced to

$$\frac{du}{dx}=\frac{du}{dx}+\frac{du}{dq}\frac{dq}{dx}+\frac{du}{dr}\frac{dr}{dx},$$

where it will be perceived that the total coefficient $\frac{du}{dx}$ in the first member of the equation, is apparently the same as the partial quotient in the second member; consequently, for distinction's sake, we inclose the partial quotient in a parenthesis, thus $\left(\frac{du}{dx}\right)$. Hence, the preceding equation will be written in the form,

$$\frac{du}{dx}=\left(\frac{du}{dx}\right)+\frac{du}{dq}\frac{dq}{dx}+\frac{du}{dr}\frac{dr}{dx};$$

and we may clearly proceed in like manner in all analogous cases.

(17.) It may not be improper, in concluding this section, to notice some of the different methods that have been used by different authors in treating the Differential Calculus.

1. Leibnitz and Newton, *the illustrious founders* of the Calculus under different forms, respectively used the *infinitesimal method, and that of the limiting ratio.*

Thus, to find the differential of x^3; we change x into $x+h$ and thence get $(x+h)^3-x^3=3x^2h+3xh^2+h^3$, for what is generally called the difference of x^3; noticing, that it is sometimes called the increment or decrement of x^3, accordingly as it is positive or negative.

If h is finite, the difference being evidently finite, is called a *finite difference;* and is often denoted by writing the Greek letter Δ (delta), called the *characteristic of finite differences,* before or to the left of x^3; and since $h = x + h - x$, we write Δx for h; consequently, for $(x+h)^3 - x^3 = 3x^2h + 3xh^2 + h^3$, we may write $\Delta x^3 = 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3$: noticing, that x^3 or (more generally) $x^3 + c$, c being constant, is often called the integral of Δx^3 or of its equivalent, $3x^2\Delta x + 3x\Delta x^2 + \Delta x^3$.

If h is unlimitedly small, or an infinitesimal, it is clear that $3x^2h + 3xh^2 + h^3$ will also be unlimitedly small, or an infinitesimal; and if infinitesimal differences, sometimes called *differentials*, are distinguished from finite differences by writing d for Δ, *then, according to the method of* Leibnitz, the equation $(x+h)^3 - x^3 = 3x^2h + 3xh^2 + h^3$ becomes $dx^3 = 3x^2dx + 3xdx^2 + dx^3$; *for which, on account of the comparative minuteness of* $3xdx^2$ and dx^3, *we may evidently write* $dx^3 = 3x^2dx$, which is of the same form, that our rule at p. 5 will give for the differential of x^3: noticing that $x^3 + c$ is called the general integral of dx^3, or of its equivalent, $3x^2dx$.

To signify that the integral of any finite difference is to be taken, the Greek letter Σ (sigma) is generally written before or to the left of the difference, inclosed in a parenthesis, if necessary. Thus, $\Sigma\Delta x^3 = \Sigma\,[3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3]$, which clearly equals $3\Sigma x^2\Delta x + 3\Sigma x(\Delta x)^2 + \Sigma(\Delta x)^3$, is used to denote that the integral of Δx^3, or of its equivalent,

$$3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3,$$

is to be taken; and since

$(x+h)^3 - x^3 = (x+h)^3 + c - x^3 - c = \Delta(x^3 + c)$, $c = \text{const.}$: the most general form of the indicated integral is $x^3 + c$.

In much the same way we indicate the integral of any proposed differential, by writing $\int$, called the sign of integration, or the *characteristic of integrals*, to the left or before the differential, as before. Thus, we have

$$\int dx^3 = \int d(x^3 + c),$$

in which $c =$ const., $= \int 3x^2dx = 3\int x^2dx = x^3 + c$: noticing, that the constant c is used for generality, or to make the integral applicable to any case that may be required.

Again, resuming $(x + h)^3 - x^3 = 3x^2h + 3xh^2 + h^3$, and dividing its members by h, it will become

$$\frac{(x+h)^3 - x^3}{h} = 3x^2 + 3xh + h^2;$$

which clearly shows if h is diminished indefinitely, the right member has $3x^2$ for its limit.

Hence, according to the common method of taking the limit, by putting $h = 0$, the equation is reduced to the form $\frac{0}{0} = 3x^2$; or since for $\frac{0}{0}$ we ought evidently to write $\frac{dx^3}{dx}$ we have $\frac{dx^3}{dx} = 3x^2dx$; see my Algebra, pages 256 and 257.

Since $(3x^2h + 3xh^2 + h^3) \div h = 3x^2 + 3xh^2 + h^2$, this quotient is often (with great impropriety) called the ratio of the increment or decrement of x^3 to the corresponding increment or decrement of the independent variable x; and $3x^2$, the limit of the quotient, is often improperly called the limit of the ratio when h is infinitesimal.

The preceding process *is substantially the same as* Newton's *method of limits*.

Because $\quad [a(x + h)^n + c - (ax^n + c)]$

$$= a\left[nx^{n-1}h + \frac{n(n-1)}{1.2}x^{n-2}h^2 +, \text{\&c.}\right]$$

is under the form of an exact difference, if h is finite, the equation (agreeably to what has been done) can be expressed by the form

$$\Delta(ax^n + c) = a\left[nx^{n-1}\Delta x + \frac{n(n-1)}{1.2}x^{n-2}(\Delta x)^2 + , \&c.\right];$$

but if h is infinitesimal, the equation is equivalent to

$$d(ax^n + c) = nax^{n-1}dx.$$

Similarly, because $(x + h)(y + k) - xy = xk + yh + hk$ is under the form of an exact difference, if h and k are finite, the equation may be expressed by the form

$$\Delta(xy + c) = x\Delta y + y\Delta x + \Delta x\Delta y;$$

but if h and k are infinitesimals, the equation becomes

$$d(xy + c) = xdy + ydx;$$

by rejecting $dxdy$ on account of its comparative minuteness.

It is manifest from these examples, that in order to find the integral of any finite difference or differential, it must be exact, or be reducible to a difference or differential which is either exact, or differs insensibly from an exact difference or differential.

2. Resuming the equation

$$(x + h)^3 - x^3 = 3x^2h + 3xh^2 + h^3,$$

and putting dx for h in the first term $3x^2h$ of the difference of x^3, it will become $3x^2dx$. If the operation to be performed on x^3, in order to obtain $3x^2dx$ from it, is denoted by $d.x^3$ or dx^3, we shall have $d.x^3 = dx^3 = 3x^2dx$; which indicates and expresses the differential of x^3, obtained by definition, *according to the method proposed by the celebrated* Lagrange.

Supposing X to be any function of x, and that X becomes X′ when x is changed into $x \pm h$; then, supposing x and h

to be undetermined, Lagrange proved that X′ may be expressed by the form $X \pm X_1 h + X_2 \frac{h^2}{1.2} \pm X_3 \frac{h^3}{1.2.3} +$, &c.; in which, he called X_1, X_2, X_3, &c., the first, second, third, &c., derived functions of X; and it is easy to perceive that the series is the same as Taylor's Theorem.

3. The difficulties and unsatisfactoriness that have attended the treatment of the first principles of the Differential Calculus, appear to us to have arisen from the circumstance, that it has been thought necessary to convert X′ into a series of the form $X + Ah + A_1h^2 + A_2h^3 +$, &c., and then to reduce the difference $X' - X = Ah + A_1h^2 + A_2h^3 +$, &c., to its first term Ah, in order to get $dX = Adx$, or the differential of X. For this process has evidently introduced the infinitesimals of Leibnitz, and the limiting ratios of Newton and others, into the Calculus, as furnishing reasons why the terms A_1h^2, A_2h^3, &c., must be rejected, in comparison to Ah. Whereas, the true reason for the omission of these terms, is that so long as x and h are indeterminates, the term Ah represents the sum of all the changes of X that result from the separate change $x' - x = h$ of each x contained in X.

And it is evident from the reasoning in (9) at p. 15, that we may consider the terms that follow the second term $\frac{dX}{dx}h$ in Taylor's Theorem, as deducible from it when x and h are regarded as being indeterminates, in a way very analogous to that of finding the terms that follow the second term from it, in the investigation of the Binomial Theorem: see Ex. 16, p. 56, of my Algebra.

SECTION II.

TRANSCENDENTAL FUNCTIONS.

(1.) WHEN a function is such that it can not be expressed by means of its variable and constants in a finite number of algebraic terms, it is called a *transcendental function*. Thus, $\log x$, a^x, $\sin x$, $\cos x$, &c., are transcendental functions: the first being a *logarithmic* function, the second an *exponential* function, and the third and fourth are *circular* functions.

(2.) Any number or quantity *may be expressed in a transcendental form*.

For if a represents any number or quantity, it is clear that for a we may write

$$\left(a-\frac{a^2}{2}+\frac{a^3}{3}-\frac{a^4}{4}+,\ \&c.\right)+\left(a-\frac{a^2}{2}+\frac{a^3}{3}-\frac{a^4}{4}+,\ \&c.\right)^2\frac{1}{1.2}$$

$$+\left(a-\frac{a^2}{2}+\frac{a^3}{3}-\frac{a^4}{4}+,\ \&c.\right)^3\frac{1}{1.2.3}+,\ \&c.;$$

in which $$a-\frac{a^2}{2}+\frac{a^3}{3}-\frac{a^4}{4}+,\ \&c.,$$

is called the *hyperbolic* or *Napierian* logarithm of $1+a$.

Hence, if we put $a-\frac{a^2}{2}+\frac{a^3}{3}-\frac{a^4}{4}+,\ \&c.=A$, we shall clearly have $1+a=1+A+\frac{A^2}{1.2}+\frac{A^3}{1.2.3}+,\ \&c.$; and in like manner, if $b-\frac{b^2}{2}+\frac{b^3}{3}-\frac{b^4}{4}+,\ \&c.$, is represented by B, we shall have $1+b=1+B+\frac{B^2}{1.2}+\frac{B^3}{1.2.3}+,\ \&c.$

(3.) The product of the corresponding members of these equations will be of a similar form.

For we shall clearly have

$$(1+a)(1+b) = 1 + a + b + ab = 1 + (A+B) + (A+B)^2 \frac{1}{1.2} + (A+B)^3 \frac{1}{1.2.3} +, \&c.$$

If $a + b + ab - \frac{(a+b+ab)^2}{2} + \frac{(a+b+ab)^3}{3} -, \&c.,$

is represented by C, it is clear from what has been done, that the preceding equation is equivalent to

$$1 + C + \frac{C^2}{1.2} + \frac{C^3}{1.2.3} +, \&c.$$

$$= 1 + A + B + (A+B)^2 \frac{1}{1.2} + (A+B)^3 \frac{1}{1.2.3} +, \&c.;$$

which clearly gives $C = A + B$.

Because A and B are the hyperbolic logarithms of $1 + a$ and $1 + b$, and that C is the hyperbolic logarithm of their product, it results from the preceding equation, *that the hyperbolic logarithm of a product equals the sum of the logarithms of its factors.*

If the members of $C = A + B$ are multiplied by the arbitrary multiplier m, called *the modulus;* it is clear that its properties will not be changed, and we shall get

$$mC = mA + mB;$$

such, that mA, mB, and mC may be called logarithms of $1 + a$, $1 + b$, and of their product.

Hence, in any system of logarithms, *the logarithm of a product equals the sum of the logarithms of its factors;* reciprocally, *the logarithm of a quotient equals the logarithm of the dividend, minus that of the divisor.*

Hence, too, *the logarithm of a power equals the logarithm of its root multiplied by the index of the power;* and reciprocally, *the logarithm of a power, divided by its index, equals the logarithm of its root.*

If the logarithm of a number or quantity, whose modulus is m, is indicated by writing log before or to the left of it (inclosed in a parenthesis when necessary), we shall clearly have $\log\ (1+a) = m\left(a - \frac{a^2}{2} + \frac{a^3}{3} - \frac{a^4}{4} +, \&c.\right) \ldots\ldots (a)$; which we shall call the *Logarithmic Theorem.*

It is evident from what has been done, that we shall have

$$(1+a)^x = 1 + Ax + \frac{(Ax)^2}{1.2} + \frac{(Ax)^3}{1.2.3} +, \&c. \ldots\ldots\ldots (b);$$

which is called the *Exponential Theorem*, in which A and Ax are the hyperbolic logarithms of $1+a$ and $(1+a)^x$.

If $A = 1$, (b) becomes

$$(1+a)^x = 1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} +, \&c. = \left(1 + 1 + \frac{1}{1.2} +, \&c.\right)^x \ldots\ldots\ldots\ldots (b');$$

which gives

$$1 + a = 1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} +, \&c. = 2.7182818284 +, \&c.$$

which is generally expressed by e, and is called the *base of hyperbolic logarithms*, since its hyperbolic logarithm is supposed to be unity or 1; consequently, putting e for $1+a$ in (b'), it becomes

$$e^x = 1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} +, \&c. \ldots\ldots\ldots\ldots (b'');$$

which shows, if we put $e^x = N$, that we shall have $x =$ the hyperbolic logarithm of N, since that of $e = 1$.

If we write log before a number or quantity (inclosed in a parenthesis if necessary) to denote its hyperbolic logarithm, it is clear that $\log\ (1+a)^x = Ax$; and as

$$\log (1 + a)^x = m \log (1 + a)^x,$$

we get $$\log (1 + a)^x = mAx.$$

If we assume $mA = 1$, or $m = \frac{1}{A}$, the preceding equation becomes $\log (1 + a)^x = x$, and of course $\log (1 + a) = 1$; consequently, $1 + a$ represents the base of the logarithms denoted by log. Hence, assuming $(1 + a)^x = N$, we have $\log (1 + a)^x = \log N = x$; since $1 + a$ is supposed to be taken for the base of the logarithms represented by log.

Because $\log N = m \log N = \frac{\log N}{A}$, *it results that we shall get* log N, *by dividing the hyperbolic logarithm of* N *by the hyperbolic logarithm of the base, or by multiplying it by the modulus* $\left(\frac{1}{A}\right)$; reciprocally, log N, *multiplied by the hyperbolic logarithm of the base, or divided by the modulus, gives the hyperbolic logarithm of* N.

Thus, if the base $1 + a = 10 =$ the base of common logarithms, the tables of hyperbolic logarithms give $\log 10 = 2.3025850929$, and thence the modulus

$$(m) = \frac{1}{A} = \frac{1}{\log 10} = 0.4342944819.$$

Again, from the tables we have $\log 2 = 0.6931471$, and thence we get $\log 2 =$ the common logarithm of 2, equals

$$\frac{0.6931471}{\log 10} = 0.6931471 \times 0.4342944 = 0.3010299,$$

which agrees with the common logarithm of 2, as given by the logarithmic tables. Reciprocally,

$$\log. 2 \times \log 10 = \log 2 \div 0.4342944 = 0.6931471,$$

equals the hyperbolic logarithm of 2.

It follows from what has been done, that the calculation of a table of logarithms to any base may be considered as being reduced to the calculation of hyperbolic logarithms. For examples in illustration of the calculation and use of logarithms in the solution of problems, the reader is referred to p. 527, &c., of my Algebra.

Resuming $e^x = 1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} +$, &c., from ($b''$), p. 51, and changing x successively into $x\sqrt{-1}$ and $-x\sqrt{-1}$, we get the equations

$$e^{x\sqrt{-1}} = 1 + x\sqrt{-1} - \frac{x^2}{1.2} - \frac{x^3\sqrt{-1}}{1.2.3} + \frac{x^4}{1.2.3.4} +, \text{\&c.}$$

$$= 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \frac{x^6}{1.2.3.4.5.6} +, \text{\&c.},$$

$$+ \left(x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} -, \text{\&c.}\right)\sqrt{-1};$$

$$e^{-x\sqrt{-1}} = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \frac{x^6}{1.2.3.4.5.6} +, \text{\&c.},$$

$$- \left(x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} -, \text{\&c.}\right)\sqrt{-1}.$$

By taking the half sum and half difference of these equations, we get

$$(e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}) \div 2 = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} -, \text{\&c., and}$$

$$(e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}) \div 2\sqrt{-1} = 1 - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} -, \text{\&c.};$$

which (in trigonometry) are called the *cosine* and *sine* of x.

Denoting the sine and cosine by writing sin and cos for them, the preceding equations may be written in the forms

$$\sin x = \frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{2\sqrt{-1}}, \text{ and } \cos x = \frac{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}}{2} \quad ..(c).$$

By adding the squares of (c), we get $\sin^2 x + \cos^2 x = 1$; which is also evident from

$$\sin x = x - \frac{x^3}{1.2.3} +, \text{\&c., and } \cos x = 1 - \frac{x^2}{1.2} +, \text{\&c.}$$

We are now prepared to show how to find the differentials of logarithmic, exponential, and circular functions.

(4.) *To show how to find the differentials of logarithmic and exponential functions.*

We will show how to find the differential of a variable or function represented by $\log x$.

From (a), given at p. 51, if we put x for $1 + a$, we must clearly put $x - 1$ for a, and we shall have

$$\log x = m \left[x - 1 - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} +, \text{\&c.}\right],$$

in which m is the modulus; consequently, by taking the differential of this, m being constant, we shall have

$$d(\log x) = m\,[1 - (x-1) + (x-1)^2 - (x-1)^3 +, \text{\&c.}] \times dx$$

$$= \frac{mdx}{1 + (x-1)} = \frac{mdx}{x};$$

and when $m = 1$, we have $d\,(\log x) = \dfrac{dx}{x}$.

Hence the differential of the logarithm of a variable or function can be found by the following

RULE.

1. Divide the differential of the variable or function by the variable or function, and the quotient, multiplied by the modulus, gives the differential.

2. If the modulus is unity, or the logarithm hyperbolic, then divide the differential by the variable or function, for the differential.

REMARKS.—1. When it is possible, hyperbolic logarithms ought always to be used in finding differentials, because their differentials are of the most simple forms.

2. It clearly results from the rule that the differential of a variable or function equals the differential of its hyperbolic logarithm multiplied by the variable or function.

EXAMPLES.

1. The differentials of $\log(a+x)$ and $\log ax = \log a + \log x$, are $\frac{mdx}{a+x}$ and $\frac{mdx}{x}$.

2. The differentials of $\log(x+y)$ and $\log\frac{x}{y} = \log x - \log y$, are $\frac{dx+dy}{x+y}$ and $\frac{dx}{x} - \frac{dy}{y} = \frac{ydx - xdy}{xy}$.

3. The differentials of $\log(a^2+x^2)$

and $$\log(a^2 - x^2) = \log(a+x) + \log(a-x),$$

are $\frac{2mxdx}{a^2+x^2}$ and $\frac{dx}{a+x} - \frac{dx}{a-x} = -\frac{2xdx}{a^2-x^2}$.

4. The differentials of

$$\log(x^2 - a^2) = \log(x+a) + \log(x-a)$$

and $$\log\frac{2}{x} = \log 2 - \log x,$$

are $$\frac{dx}{x+a} + \frac{dx}{x-a} = \frac{2xdx}{x^2-a^2},$$

which is the same as to divide $d(x^2-a^2)$ by (x^2-a^2), and $-\frac{dx}{x}$.

5. The differentials of $\log\sqrt[n]{(a^2+x^2)^{m'}} = \log(a^2+x^2)^{\frac{m'}{2}}$ and $\log[x+\sqrt{(x^2\pm a^2)}]$ are $\frac{2m'xdx}{(a^2+x^2)}$,

and $dx\left(1+\frac{x}{\sqrt{(x^2\pm a^2)}}\right) \div [x+\sqrt{(x^2\pm a^2)}] = \frac{dx}{\sqrt{(x^2\pm a^2)}}$.

6. The differential of $\log \frac{\sqrt{(a^2+x^2)}-a}{\sqrt{(a^2+x^2)}+a}$, is

$$\frac{xdx}{\sqrt{(a^2+x^2)}}\left(\frac{1}{\sqrt{(a^2+x^2)}-a}-\frac{1}{\sqrt{(a^2+x^2)}+a}\right)=\frac{2adx}{x\sqrt{(a^2+x^2)}}$$

7. The differential of ax^m is $ax^m \times \frac{mdx}{x} = max^{m-1}dx.$

8. The differential of xy is

$$xy \times d \log xy = xyd\,(\log x + \log y)$$

$$= xy\left(\frac{dx}{x}+\frac{dy}{y}\right) = ydx + xdy.$$

9. The differential of $\frac{x}{y}$, is $\frac{x}{y}\left(\frac{dx}{x}-\frac{dy}{y}\right)=\frac{ydx-xdy}{y^2}.$

10. The differential of a^x is

$$a^x \times d(\log a^x) = a^x d(\log a \times x) = a^x \log a \times dx;$$

which can be also found from assuming $y = a^x$, or $\log y = x \log a$, or $\frac{dy}{y} = \log a dx$, or $dy = y \log a dx = a^x \log a dx$, as before.

It is hence evident, that when the exponent of an exponential is alone variable, we can find the differential of the exponential by the following

RULE.

Multiply the hyperbolic logarithm of the base or root of the exponential by the exponential, and the product by the differential of the variable exponent.

REMARK.—If the base of the exponential is also variable, then we must add the differential, regarding the base as alone variable to the preceding differential; and the result will be the complete differential, when the base and exponent of the exponential are variable.

EXAMPLES.

1. The differentials of 2^x and 3^y, are $2^x \times \log 2 \times dx$, and $3^y \log 3 \times dy$: noticing that 2 and 3 are the constant roots of the exponentials, whose variable exponents are x and y.

2. The differentials of e^x and e^{-x}, are $e^x dx$ and $-e^{-x}dx$; since $\log e = 1$.

3. The differentials of ba^x and c^{ax}, are $ba^x \log a \times dx$, and $c^{ax} \log c \times adx$.

4. The differentials of $e^{\log x}$ and $a^{\log x}$, are

$$e^{\log x}\frac{dx}{x} \quad \text{and} \quad a^{\log x} \log a \times \frac{dx}{x}.$$

5. The differentials of ay^x and y^{-x}, are

$$xay^{x-1}dy + ay^x \log y \times dx$$

and

$$-xy^{-x-1}dy - y^{-x} \log ydx,$$

as is clear from the rule and remark.

6. The differentials of a^{e^x} and e^{a^x}, are $a^{e^x} \log ae^x dx$ and $e^{a^x} \log a \times a^x dx$; noticing that e^x and a^x are variable exponents of a and e, and that e stands for the hyperbolic base.

7. The differentials of z^{x^y} and $(\log x)^{\log x}$, are

$$z^{x^y} \log z \times (yx^{y-1}\,dx + x^y \log xdy) + x^y z^{x^y - 1} dz,$$

and $(\log x)^{\log x} \times \log (\log x)\dfrac{dx}{x} + \log x \times (\log x)^{\log x - 1} \times \dfrac{dx}{x}$

$$= (\log x)^{\log x} \times (1 + \log^2 x)\frac{dx}{x}:$$

noticing, that the notation $\log^2 x$ is used for $\log (\log x)$, and we may also represent $\log [\log (\log x)]$ by writing $\log^3 x$; and so on, to any extent.

8. The differentials of $e^{\sqrt{(a^2-x^2)}}$ and $e^{\log^2 x}$, are

$$e^{\sqrt{(a^2-x^2)}} \times -\frac{xdx}{\sqrt{(a^2-x^2)}} = -e^{\sqrt{(a^2-x^2)}}\frac{xdx}{\sqrt{(a^2-x^2)}}$$

and $$e^{\log^2 x}\frac{dx}{x\log x}.$$

9. The differentials of $e^{x\sqrt{-1}}$ and $e^{-x\sqrt{-1}}$, are

$$e^{x\sqrt{-1}} \times \sqrt{-1}\,dx \text{ and } e^{-x\sqrt{-1}} \times -\sqrt{-1}\,dx.$$

10. The differentials of $a^{x\sqrt{-1}}$ and $a^{-x\sqrt{-1}}$, are

$a^{x\sqrt{-1}}\log a \times dx\sqrt{-1}$, and $a^{-x\sqrt{-1}}\log a \times -dx\sqrt{-1}$.

(5.) *We will now show how to find the differentials of circular functions.*

From (*c*) page 53, we have $\sin x = \dfrac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{2\sqrt{-1}}$

and $$\cos x = \frac{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}}{2};$$

whose differentials give

$$d(\sin x) = \frac{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}}{2}dx = \cos x dx,$$

and $$d(\cos x) = \frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{2} \times \sqrt{-1}\,dx$$

$$= -\frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{2\sqrt{-1}}dx = -\sin x dx.$$

By adding the squares of these differentials, we shall get

$$(d\sin x)^2 + (d\cos x)^2 = \cos^2 x dx^2 + \sin^2 x dx^2$$
$$= (\cos^2 x + \sin^2 x)\,dx^2 = dx^2,$$

since we have shown, at page 54, that $\cos^2 x + \sin^2 x = 1$.

REMARK.—It is clear from the expressions for $\sin x$ and $\cos x$, that they, together with x and dx, represent numbers or geometrical ratios, and not quantities.

It clearly follows from what has been done, that we can find the differentials of the sine and cosine of any variable by the following

RULES.

1. The differential of the sine equals the cosine multiplied by the differential of the variable.

2. The differential of the cosine equals minus the product of the sine and the differential of the variable.

EXAMPLES.

1. The differentials of $\sin 2x$ and $\cos 2x$, are $2 \cos 2x dx$ and $-2 \sin 2x dx$.

2. The differentials of $\sin mx$ and $\cos mx$, are

$$m \cos mx dx \quad \text{and} \quad -m \sin mx dx.$$

3. The differentials of $\sin (a \pm x)$ and $\cos (a \pm x)$, are

$$\pm \cos (a \pm x)\, dx \quad \text{and} \quad \mp \sin (a \pm x)\, dx.$$

4. The differentials of $\sin^2 x$ and $\cos^2 x$, are

$$2 \sin x \cos x dx \quad \text{and} \quad -2 \sin x \cos x dx.$$

5. The differentials of $\sin^m x$ and $\cos^m x$ are

$$m \cos x \sin^{m-1} x dx \quad \text{and} \quad -m \sin x \cos^{m-1} x dx:$$

noticing, that the exponent m denotes the mth powers of $\sin x$ and $\cos x$.

6. The differentials of $\tan x = \frac{\sin x}{\cos x}$ and $\cot x = \frac{\cos x}{\sin x}$, are

$$d \tan x = \frac{d \sin x \times \cos x - d \cos x \times \sin x}{\cos^2 x}$$

$$= \frac{\cos^2 x dx + \sin^2 x dx}{\cos^2 x} = \frac{dx}{\cos^2 x} = \sec^2 x dx$$

and $$d \cot x = \frac{d \cos x \times \sin x - d \sin x \times \cos x}{\sin^2 x}$$

$$= -\frac{dx}{\sin^2 x} = -\operatorname{cosec}^2 x dx.$$

Hence, *the differential of the tangent of a variable, equals the differential of the variable divided by the square of its cosine or multiplied by the square of its secant; since unity divided by the cosine is (in Trigonometry) called the secant.*

And *the differential of the cotangent of a variable, equals minus the differential of the variable divided by the square of its sine or multiplied by the square of its cosecant.*

EXAMPLES.

1. The differentials of $\tan 2x$ and $\cot 2x$, are

$$\frac{2dx}{\cos^2 2x} \quad \text{and} \quad -\frac{2dx}{\sin^2 2x}.$$

2. The differentials of $\tan mx$ and $\cot mx$, are

$$\frac{mdx}{\cos^2 mx} \quad \text{and} \quad -\frac{mdx}{\sin^2 mx}.$$

3. The differentials of $\tan (a \pm x)$ and $\cot (a \pm x)$, are

$$\frac{\pm dx}{\cos^2 (a \pm x)} \quad \text{and} \mp \frac{dx}{\sin^2 (a \pm x)}.$$

4. The differentials of $\tan x^m$ and $\cot x^m$, are

$$\frac{m \tan x^{m-1} dx}{\cos^2 x} \quad \text{and} \quad -\frac{m \cot x^{m-1} dx}{\sin^2 x}$$

5. It is easy to perceive that we may, in much the same way, find the differentials of $\frac{1}{\cos x}$ and $\frac{1}{\sin x}$; which give

$$d\frac{1}{\cos x} = -\frac{d \cos x}{\cos^2 x} = \frac{\sin x dx}{\cos^2 x} = \tan x \sec x dx;$$

and in like manner

$$d\frac{1}{\sin x} = -\frac{d \sin x}{\sin^2 x} = -\cot x \operatorname{cosec} x dx.$$

Because $\frac{1}{\cos x}$ and $\frac{1}{\sin x}$ are called the secant and cosecant of x, we hence find the differentials of the secant and cosecant of any variable, by the following

RULE.

The differential of the secant of a variable equals the product of the tangent, secant, and differential of the variable.

And, *the differential of the cosecant of a variable equals minus the product of the cotangent, cosecant, and differential of the variable.*

Thus, the differentials of $\sec^m x$ and $\operatorname{cosec}^m x$, are

$$\tan x \sec x \times m \sec^{m-1} x dx$$

and $$-\cot x \operatorname{cosec} x \times m \operatorname{cosec}^{m-1} \; dx;$$

and the differentials of $\sec(a^m + x^m)$ and $\operatorname{cosec}(a^m - x^m)$, are

$$\tan(a^m + x^m) \sec(a^m + x^m) \times mx^{m-1} dx,$$

$$-\cot(a^m - x^m) \operatorname{cosec}(a^m - x^m) \times mx^{m-1} dx.$$

6. Because versin $x = 1 - \cos x$ and coversin $x = 1 - \sin x$, their differentials are $\sin x dx$ and $-\cos x dx$; *which are the same as those of the cosine and sine after their signs are changed.*

7. Since suversine of $x = 1 + \cos x$, and cosuversine $x = 1 + \sin x$, their differentials are $-\sin x dx$ and $\cos x dx$; *which are the same as those of the cosine and sine.*

8. The differentials of $\sin(\sin x)$ and $\cos(\sin x)$, are evidently $$d \sin(\sin x) = \cos(\sin x) \cos x dx,$$

and $$d \cos(\sin x) = -\sin(\sin x) \cos x dx;$$

and the differentials of $\sin(\cos x)$ and $\cos(\cos x)$, are

$$d \sin(\cos x) = -\cos(\cos x) \sin x dx,$$

and $$d \cos(\cos x) = \sin(\cos x) \sin x dx;$$

and so on, for other analogous forms.

9. The differentials of log sin x and log cos x, are

$$d \log \sin x = \frac{d \sin x}{\sin x} = \cot x dx,$$

and
$$d \log \cos x = \frac{d \cos x}{\cos x} = -\tan x dx;$$

and these multiplied by the modulus (m), will give the differentials of log sin x and log cos x.

10. The differentials of log tan x and log cot x, are

$$d . \log \tan x = \frac{d . \tan x}{\tan x} = \frac{dx}{\cos^2 x \tan x} = \frac{dx}{\sin x \cos x} = \frac{2dx}{\sin 2x},$$

and
$$d . \log \cot x = \frac{d . \cot x}{\cot x} = -\frac{dx}{\sin x \cos x} = -\frac{2dx}{\sin 2x};$$

and we may proceed in like manner in all analogous cases.

(6.) Since, to find the preceding values, it is necessary to know those of sin x, cos x, tan x, &c., when x and dx are given; we will now show how to obtain their values to any degree of exactness that may be required, by converging series.

To the end in view, we will find the expansions of sin $(x \pm h)$ and cos $(x \pm h)$, when arranged according to the ascending powers of h.

Thus, if we put sin x for X, and sin $(x \pm h)$ for X′, and $\pm h$ for h, in Taylor's Theorem, or (a), given at p. 16, we shall have $\frac{dX}{dx} = \frac{d \sin x}{dx} = \cos x$, $\frac{d^2X}{dx^2} = \frac{d \cos x}{dx} = -\sin x$, $\frac{d^3X}{dx^3} = -\cos x$, $\frac{d^4X}{dx^4}$ sin x, and so on.

Hence, from the substitution of the preceding values in (a), we get, after duly ordering the terms,

$$\sin(x \pm h) = \sin x \left(1 - \frac{h^2}{1.2} + \frac{h^4}{1.2\text{-}3.4} -, \&c.\right)$$

$$\pm \cos x \left(h - \frac{h^3}{1.2.3} + \frac{h^5}{1.2.3.4.5} -, \&c.\right)$$

$$= \sin x \cos h \pm \cos x \sin h \text{ (see p. 53)} \quad . \ . \ . \ . \quad (d).$$

In like manner, we easily get

$$\cos(x \pm h) = \cos x \left(1 - \frac{h^2}{1.2} + \frac{h^4}{1.2.3.4} -, \&c.\right)$$

$$\mp \sin x \left(h - \frac{h^3}{1.2.3} + \frac{h^5}{1.2.3.4.5} -, \&c.\right)$$

$$= \cos x \cos h \mp \sin x \sin h \text{ (p. 53)} \quad . \ . \ . \ . \quad (d').$$

If we put $h = x$, and use the upper signs in these formulæ, they give

$$\sin 2x = 2 \sin x \cos x, \quad \text{and} \quad \cos 2x = \cos^2 x - \sin^2 x;$$

or, since $\sin^2 x + \cos^2 x = 1$, we have $\cos^2 x = 1 - \sin^2 x$, which reduces

$$\cos 2x = \cos^2 x - \sin^2 x \quad \text{to} \quad \cos 2x = 1 - 2 \sin^2 x;$$

which, by changing x into $\frac{x}{2}$, becomes $\cos x = 1 - 2 \sin^2 \frac{x}{2}$.

As an example of the use of the last of these formulæ, we shall successively put $x = 1.5$ and 1.6, and thence get $\frac{x}{2} = 0.75$, and $\frac{x}{2} = 0.8$, for the corresponding values of $x \div 2$.

From the substitution of these values in

$$\sin \frac{x}{2} = \frac{x}{2} - \frac{\left(\frac{x}{2}\right)^3}{1.2.3} + \frac{\left(\frac{x}{2}\right)^5}{1.2.3.4.5} - \ \&c.,$$

we shall get

$$\sin 0.75 = 0.75 - 0.070312 + 0.001977 - 0.000026 +, \&c., = 0.681639,$$

and

$$\sin 0.8 = 0.8 - 0.085333 + 0.002730 - 0.000041 +, \&c., = 0.717356.$$

Hence, we get $2 \sin^2 \frac{x}{2} = 2 \sin^2 0.75 = 0.929268$, and thence we have

$$\cos x = \cos 1.5 = 1 - 0.929268 = 0.070732.$$

In a similar way, we have $2 \sin^2 0.8 = 1.029059$, and thence

$$\cos 1.6 = 1 - 1.029059 = -0.029059.$$

Hence, there is clearly a value of x greater than 1.5 and less than 1.6, which we shall represent by $\frac{\pi}{2}$, such that we shall have $\cos \frac{\pi}{2} = 0$; and thence from $\cos^2 \frac{\pi}{2} + \sin^2 \frac{\pi}{2} = 1$, we get

$$\sin^2 \frac{\pi}{2} = 1, \quad \text{or} \quad \sin \frac{\pi}{2} = \pm 1.$$

From $\cos x = 1 - 2 \sin^2 \frac{x}{2} = 1 - 2 \left\{ \left(\frac{x}{2}\right) - \frac{\left(\frac{x}{2}\right)^3}{1.2.3} +, \&c. \right\}^2$,

by putting $x = 0$, we also have $\cos 0 = 1$, and $\sin \frac{x}{2} = 0$, or $\sin x = 0$; and from

$$\cos \pi = \cos^2 \frac{\pi}{2} - \sin^2 \frac{\pi}{2}, \text{ since } \cos^2 \frac{\pi}{2} = 0, \text{ and } \sin^2 \frac{\pi}{2} = 1,$$

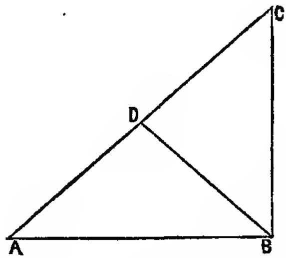
we have

$$\cos \pi = -1.$$

(7.) For convenience in what is to follow, *we now propose to show how to represent* 1, sin x, *and* cos x, *geometrically.*

Since $\sin^2 x + \cos^2 x = 1$, it is clear that the sum of no two of the three, 1, sin x, cos x, can be less than the third, while their difference can not be greater than the third.

These results correspond to well-known properties of the sides of a rectilineal triangle; viz., that the sum of any two of its sides can not be less than the third, while their difference can not be greater; properties, that evidently follow from the consideration, that a straight line is the shortest distance between its extremities.



Thus, let ABC denote a rectilineal triangle; such, that AC equals any unit of length, while CB and AB are the same parts of AC that $\sin x$ and $\cos x$ are of 1; then, it is clear that AC, CB, and AB, may be taken as representatives of 1, $\sin x$, and $\cos x$. Similarly, if we take the equation $\sin^2 x' + \cos^2 x' = 1$, such that $\sin x'$ is the same part of 1 that AB is of AC, it is manifest that 1, $\sin x'$, and $\cos x'$, will also be represented by AC, AB, and BC; consequently, $\sin x = \cos x'$, and $\cos x = \sin x'$.

From (d), page 63, $\sin (x + h) = \sin x \cos h + \cos x \sin h$; which, by putting x' for h, becomes

$\sin (x + x') = \sin x \cos x' + \cos x \sin x' = \sin^2 x + \cos^2 x = 1$; since $\cos x' = \sin x$ and $\sin x' = \cos x$, and that

$$\sin^2 x + \cos^2 x = 1.$$

Because $\sin (x + x') = 1$, and that at page 64 we have $\sin \frac{\pi}{2} = 1$, it follows that we must have $x + x' = \frac{\pi}{2}$, in which

$\frac{\pi}{2}$ is a value of $x + x'$ that lies between 1.5 and 1.6. (See page 64.)

Because (page 53) $CB = \sin x = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5}$, &c.,

and that $AB = \sin x' = x' - \frac{x'^3}{1.2.3} + \frac{x'^5}{1.2.3.4.5}$, &c.,

by adding these we have

$$CB + AB = \sin x + \sin x'$$

$$= x + x' - \frac{x^3 + x'^3}{1.2.3} + \frac{x^5 + x'^5}{1.2.3.4.5} -, \&c.$$

$$= (x + x')\left(1 - \frac{x^2 - xx' + x'^2}{1.2.3} +, \&c.\right),$$

which is greater than the side AC; and if $x = 0$, $\sin x' = AC$, or if $x' = 0$, $\sin x = AC$. Since

$$AC = 1 = \sin(x + x') = (x + x')\left(1 - \frac{x^2 + 2xx' + x'^2}{1.2.3} +, \&c.\right),$$

we hence get

$$(x + x')\left(1 - \frac{x^2 - xx' + x'^2}{1.2.3} +, \&c.\right)$$

$$> (x + x')\left(1 - \frac{x^2 + 2xx' + x'^2}{1.2.3} +, \&c.\right).$$

It hence follows that x and x' must be represented by the angles A and C; for if $\sin x = 0$ we have x and the angle A each equal to naught, and AC coincides with AB; and in like manner if $\sin x' = 0$, AC coincides with BC. Because of the inequality $AC + BC > AB$, if X represents the angle B, it is clear that we must have

$$X + x - \frac{X^3 + x^3}{1.2.3} +, \&c.$$

$$= (X + x)\left(1 - \frac{X^2 - Xx + x^2}{1.2.3} +, \&c.\right) > x' - \frac{x'^3}{1.2.3} +, \&c.$$

for the proper representation of the inequality; consequently, AC being expressed in terms of X in a way similar to the representations of the other sides in terms of their opposite angles, it clearly follows that AC must be the sine of $X = \sin ABC$.

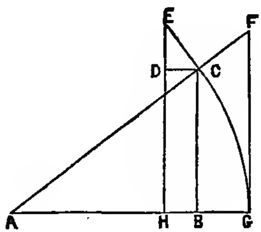
From $AC = x + x' - \frac{(x + x')^3}{1.2.3} +$, &c., $= X - \frac{X^3}{1.2.3} +$, &c., we must have X or the angle ABC equal to $x + x'$, the sum of the angles A and C. Hence (see figure), if from the right angle B, we draw the right line BD meeting AC in D, so as to make the angle CBD = the angle C, we shall have the angle ABD = the angle A.

Hence, the lines AD, DB, and DC, are equal, and the points A, B, C, lie in the circumference of a circle whose center is D and radius DB. If the angles A and C equal each other, it is clear that AB = BC, and of course $AC^2 = AB^2 + BC^2$ or $4AD^2 = 2AB^2$ or $AB^2 = 2AD^2 = AD^2 + BD^2$; consequently, in the triangle ADB the angle D equals the sum of the remaining angles of the triangle. But, since the triangles ADB and CDB are clearly identical, it results that their angles at D must equal each other, and of course from the well-known definition of a right angle, each of them is a right angle. Hence, the angles at A and B in the triangle ADB are together equivalent to a right angle; and in the triangle CDB, the sum of the angles at C and B is equivalent to a right angle. Hence, the sum of the angles of the triangle ABC is equivalent to two right angles, and because the angle B equals the sum of the angles A and C, it is clear that B is a right angle, and that the sum of the angles A and C is equal to a right angle; and because the angles A and C make the same sum, whether they are equal or unequal, it clearly follows that their sum is always a right angle.

Also, because the angle B is always in a semicircle whose center is D and diameter AC, it follows that the angle inscribed in a semicircle is always a right angle.

Reciprocally, if one angle of a triangle is right, the sum of the other two angles is right, and the square of the numerical value of the side opposite to the right angle equals the sum of the squares of the numerical values of the other two sides. For ABC (see fig.) being the triangle, a circle described on AC as a diameter, must evidently pass through the right angle, and the triangle coincides with one of the triangles that have been considered; and thence the truth of the proposition is manifest. It may be added that the sum of the three angles of any rectilineal triangle is easily shown to be equal to two right angles.

(8.) We now propose to show how to find the numerical values of angles.



Resuming the right triangle ABC from p. 65, we have, according to what is there supposed, AC to represent any arbitrary unit of length, while the angles A and C are represented by x and x', and CB = sin x, AB = sin x' = cos x. If from A as a center, with AC as a radius, the arc CG is described meeting AB produced in G, it will represent the value of sin x. By taking the differentials of

CB = sin x and AB = cos x, we shall (as at p. 61) have d. CB = cos xdx and d. AB = − sin xdx, which give (as at p. 58) $\sqrt{[(d.\mathrm{CB})^2 + (d.\mathrm{AB})^2]} = dx$, supposing x and sin x to increase while cos x decreases. If from B toward A, BH is set off to represent d. AB = − sin xdx and HE drawn parallel to CB, meeting the tangent to the arc CG at C in E; and if through C, CD is drawn parallel to AB, meeting HE in D; then, EC represents dx, and ED = d.CB = cos xdx. For the right triangles ACB and ECD give the proportions

AC or 1 : EC :: cos x : ED, and 1 : EG :: sin x : CD = BH,

which give DE = EC × cos x, and BH = EC × sin x.

Since (neglecting the signs) BH = sin xdx, the second of these equations gives EC × sin x = sin xdx, or dx = EC; consequently, the first becomes DE = cos xdx, as it ought to be. Because the arc GC and the angle x commence together at G, and increase together from G toward C, and that the increase of the arc at any point is clearly in the direction of the tangent (at the point), CE evidently represents the differential of the arc GC; consequently, since dx = GE, it follows that dx represents the differential of the arc GC, and, of course, x equals GC; agreeably to what has been supposed.

REMARKS.—1. It is easy to perceive that we may proceed in much the same way as above, to find the differential of any proposed arc of any plane curve, by expressing it, in terms of the differentials of its rectangular co-ordinates, like AB and CB; that is, by taking the square root of the sum of the squares of their differentials at any point of the curve, for the differential of the curve at the same point.

2. In our reasonings we have, and shall, generally, take it

for granted that the reader is familiar with the definitions and leading principles of Geometry and Trigonometry. Thus, in the figure, supposed to be constructed, AC, CB, AB, are called the radius, sine, and cosine of the arc GC; also, AG, GF, and AF, are called the radius, tangent, and secant of the same arc.

3. AC being represented by 1, since the equiangular triangles ABC and AGF give the proportion

$$\text{AB} : \text{BC} :: \text{AG} : \text{GF} = t, \quad \text{or} \quad \cos x : \sin x :: 1 : t,$$

or its equivalent, $\sin x = t \cos x$,

in which $t =$ the tangent of x. Since

$$\sin x = x - \frac{x^3}{1.2.3}, \text{ \&c.,} \quad \cos x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} -, \text{ \&c.};$$

consequently, the preceding equation may be written in the form,

$$x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} -, \text{\&c.} = t\left(1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} -, \text{\&c.}\right);$$

which clearly shows that x can be expressed in a series of the odd integral powers of t.

For a simple inspection of the terms shows t to be the first term of the series; and to get the second term, we put $t + \text{A}t^3$ for x, and thence have

$$t + \text{A}t^3 - \frac{t^3}{1.2.3} +, \text{\&c.} = t - \frac{t^3}{1.2} +, \text{\&c.};$$

consequently, if we determine A, on the supposition that the terms involving t^3 destroy each other, we shall have

$$\text{A}t^3 - \frac{t^3}{1.2.3} = -\frac{t^3}{1.2}, \quad \text{or} \quad \text{A} = -\frac{1}{1.2} + \frac{1}{1.2.3} = -\frac{1}{3}.$$

If $t - \dfrac{t^3}{3} + \text{A}t^5$ is put for x, we shall in like manner get $\text{A} = \dfrac{1}{5}$; and so on. Hence, we shall have

$$x = t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} +, \text{\&c.} \quad \ldots\ldots \quad (e);$$

which is a very useful formula for finding the circumference of a circle.

Thus, if x is the numerical value of half a right angle, since $x = x'$, we have

$$\sin x = \cos x, \text{ and, of course, } t = \frac{\sin x}{\cos x} = 1;$$

consequently (since $\frac{\pi}{2}$ expresses the numerical value of a right angle), by putting 1 for t and $\frac{\pi}{4}$ for x, we shall have

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} +, \text{\&c.}$$

Again, if x is one-third of a right angle, we shall have $x' = 2x$, and $\cos x = \sin x' = \sin 2x = 2 \sin x \cos x$,

or $\sin x = \frac{1}{2}$, and thence $\cos x = \frac{\sqrt{3}}{2}$;

consequently, from $t = \frac{\sin x}{\cos x}$ we get $t = \frac{1}{\sqrt{3}}$

From the substitution of this value of t in (e), we clearly get $\frac{\pi}{6} = \frac{1}{\sqrt{3}} \left(1 - \frac{1}{3.3} + \frac{1}{5.3^2} - \frac{1}{7.3^3} + \frac{1}{9.3^4} -, \text{\&c.}\right)$, which will enable us to find the numerical value of π to any required degree of exactness.

The value of π to eight decimal places is easily found to be 3.14159265; which is clearly the numerical value of two right angles, or the semicircumference of a circle whose radius is the unit of length; consequently, the product of π and R, the radius of any other circle, gives Rπ for the length of the semicircumference of the circle whose radius is R.

For series of more rapid convergency than the above, the student is referred to page 70, volume 1, of Lacroix's "Calcul Différentiel," and to page 797 of Rutherford's edition of "Hutton's Mathematics."

(9.) We will now show how to find the differential of an arc regarded as a function of its sine, cosine, etc.; which are sometimes called *inverse functions.*

1. If $\sin z = y$ and $\cos z = x$, we get from what is done at page 58, $\cos z dz = dy$ and $\sin z dz = -dx$, or (since $\sin^2 z + \cos^2 z = 1$) $dz = \dfrac{dy}{\sqrt{1-y^2}}$ and $dz = -\dfrac{dx}{\sqrt{1-x^2}}$;

and in like manner, if we put $\tan z = t$ and $\cot z = t'$, we get, from page 59,

$$\frac{dz}{\cos^2 z} = dt \quad \text{and} \quad \frac{dz}{\sin^2 z} = -dt';$$

or
$$dz = \cos^2 z dt \quad \text{and} \quad dz = -\sin^2 z dt',$$

which are equivalent to

$$dz = \frac{dt}{1+t^2} \quad \text{and} \quad dz = -\frac{dt'}{1+t'^2}.$$

Also, if $\sec z = s$ and $\operatorname{cosec} z = s'$, we get, from what is shown at page 61,

$$\tan z \sec z dz = ds \quad \text{and} \quad \cot z \operatorname{cosec} z dz = -ds';$$

which, from $\tan z = \sqrt{s^2-1}$ and $\cot z = \sqrt{s'^2-1}$, are reducible to

$$dz = \frac{ds}{s\sqrt{s^2-1}} \quad \text{and} \quad dz = -\frac{ds'}{s'\sqrt{s'^2-1}}.$$

In much the same way, from page 61, if we put $\operatorname{versin} z = 1 - \cos z = v$ and $\operatorname{coversin} z = 1 - \sin z = v'$, we get

$$\sin z dz = dv \quad \text{and} \quad \cos z dz = -dv',$$

or $$dz = \frac{dv}{\sin z} \quad \text{and} \quad dz = -\frac{dv'}{\cos z};$$

consequently, since $\cos z = 1 - v$ and $\sin z = 1 - v'$, we get

$$dz = \frac{dv}{\sqrt{2v - v^2}} \quad \text{and} \quad dz = -\frac{dv}{\sqrt{2v' - v'^2}}.$$

2. It is manifest that the radius of the arc in the preceding formula is 1, or unity, which may easily, from the principles of homogeneity in the members of the equations, be reduced to an arc whose radius is r, after the following manner: Thus, for y^2 and x^2 in the first two equations, write

$$\frac{y^2}{r^2} \quad \text{and} \quad \frac{x^2}{r^2}$$ and they become

$$dz = \frac{dy}{\sqrt{1 - \frac{y^2}{r^2}}} \quad \text{and} \quad dz = -\frac{dx}{\sqrt{1 - \frac{x^2}{r^2}}},$$

which are easily reduced to

$$dz = \frac{rdy}{\sqrt{r^2 - y^2}} \quad \text{and} \quad dz = -\frac{rdx}{\sqrt{r^2 - x^2}};$$

and in like manner the remaining equations become

$$dz = \frac{r^2dt}{r^2 + t^2}, \qquad dz = -\frac{r^2dt'}{r^2 + t'^2},$$

$$dz = \frac{r^2ds}{s\sqrt{s^2 - r^2}}, \qquad dz = -\frac{r^2ds'}{s'\sqrt{s'^2 - r^2}},$$

$$dz = \frac{rdv}{\sqrt{2rv - v^2}}, \qquad dz = -\frac{rdv'}{\sqrt{2rv' - v'^2}},$$

which are adapted to the arc z whose radius is r.

REMARKS.—1. Differentials that are not of the preceding forms, can often be reduced to them. Thus

$$\frac{dx}{\sqrt{25 - 16x^2}} \text{ is equivalent to } \frac{\frac{5}{4}\,dx}{5\sqrt{\left(\frac{5}{4}\right)^2 - x^2}};$$

which is the differential of a circular arc whose radius is $\frac{5}{4}$ and sin $= x$ divided by 5. In like manner the differential $\frac{dv}{a^2 + b^2v^2}$ is reducible to $\frac{\frac{b}{a}\,dv}{ab\left(1 + \frac{b^2}{a^2}v^2\right)}$; which is the differential of a circular arc whose radius $= 1$ and tangent $= \frac{b}{a}\,v$, divided by ab.

2. In like manner, differentials can often be reduced to those of known logarithmic forms. Thus the differential $\frac{-2xdx}{a^2 - x^2}$ is reducible to the known logarithmic differentials

$$\frac{dx}{a + x} + \frac{-\,dx}{a - x}, \quad \text{and} \quad \frac{2adx}{x^2 - a^2}$$

is equivalent to
$$\frac{dx}{x - a} - \frac{dx}{x + a},$$

which are differentials of well-known logarithmic forms.

(10.) We will conclude this section by noticing some of the more important properties of the expressions

$$e^{x\sqrt{-1}} = \cos x + \sin x\,\sqrt{-1} \text{ and } e^{-x\sqrt{-1}} = \cos x - \sin x\,\sqrt{-1},$$

or their equivalents

$$\cos x = \frac{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}}{2} \quad \text{and} \quad \sin x = \frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{2\sqrt{-1}};$$

see page 53.

It is manifest that for the first two of these forms, we may take $e^{\pm x\sqrt{-1}} = \cos x \pm \sin x\,\sqrt{-1}$; by using the upper signs (in the ambiguous signs) for the first, and the lower signs for the second.

If mx is put for x, we shall have

$$e^{\pm mx\sqrt{-1}} = \cos mx \pm \sin mx \sqrt{-1},$$

or, because

$$e^{\pm mx\sqrt{-1}} = (e^{\pm x\sqrt{-1}})^m = (\cos x \pm \sin x \sqrt{-1})^m,$$

we shall get

$$(\cos x \pm \sin x \sqrt{-1})^m = \cos mx \pm \sin mx \sqrt{-1} \ldots (f);$$

which is called *De Moivre's Formulæ.*

Expanding the first member of this equation according to the ascending powers of $\pm \sin x \sqrt{-1}$ by the Binomial Theorem, and equating the real and imaginary parts of the members of the resulting equation, separately, we readily get

$$\cos mx = \cos^m x - \frac{m(m-1)}{1.2} \cos^{m-2} x \sin^2 x$$

$$+ \frac{m(m-1)(m-2)(m-3)}{1.2.3.4} \cos^{m-4} x \sin^4 x -, \&c.,$$

and

$$\sin mx = m \sin x \left(\cos^{m-1} x - \frac{(m-1)(m-2)}{2.3} \cos^{m-3} \sin^2 x\right.$$

$$\left. + \frac{(m-1)(m-2)(m-3)(m-4)}{2.3.4.5} \times \cos x^{m-5} \sin^4 x -, \&c.\right)$$

If in these equations we successively put $m = 2$, $m = 3$, &c., we get

$$\cos 2x = \cos^2 x - \sin^2 x = \cos^2 x - (1 - \cos^2 x) = 2\cos^2 x - 1,$$

$$\sin 2x = 2 \sin x \cos x,$$

$$\cos 3x = \cos^3 x - 3 \cos x \sin^2 x = \cos^3 x - 3 \cos x (1 - \cos^2 x)$$
$$= 4 \cos^3 x - 3 \cos x,$$

$$\sin 3x = 3 \sin x \cos^2 x - \sin^3 x = 3 \sin x - 4 \sin^3 x,$$

and so on.

If in the expressions for $\cos x$ and $\cos mx$, $\sin x$ and $\sin mx$, we put $e^{x\sqrt{-1}} = y$, and of course $e^{-x\sqrt{-1}} = \frac{1}{y}$, then we get $2 \cos x = y + \frac{1}{y}$, $2 \cos mx = y^m + \frac{1}{y^m}$;

$2 \sin x \sqrt{-1} = y - \frac{1}{y}$, and $2 \sin mx \sqrt{-1} = y^m - \frac{1}{y^m}$.

Supposing m to be a positive integer; by raising the members of $2 \cos x = y + \frac{1}{y}$ to the mth power, and uniting the first and last terms, the second and last but one terms, and so on; we shall evidently have

$$2^m \cos^m x = \left(y^m + \frac{1}{y^m}\right) + m\left(y^{m-2} + \frac{1}{y^{m-2}}\right)$$

$$+ \frac{m(m-1)}{1.2}\left(y^{m-4} + \frac{1}{y^{m-4}}\right) +, \&c.$$

If m is an odd number, since $y^m + \frac{1}{y^m} = 2 \cos mx$, $y^{m-2} + \frac{1}{y^{m-2}} = 2 \cos (m-2)x$, and so on, we readily get $2^{m-1} \cos mx = \cos mx + m \cos (m-2)x + \frac{m(m-1)}{1.2} \cos (m-4)x +, \&c.$, until the number of terms $= \frac{m+1}{2}$.

When m is an even number, we have $2^{m-1} \cos^m x = \cos mx + m \cos (m-2)x + \frac{m(m-1)}{1.2} \cos (m-4)x +, \&c.$, until there are $\frac{m}{2}$ terms containing cosines; to which must be added the term $\frac{1}{2} \frac{m(m-1) \times \ldots \times \left(\frac{m}{2}+1\right)}{1.2 \times \ldots\ldots \times \frac{m}{2}}$.

If in these formulæ we put 1, 2, 3, &c., successively, for m, we readily get the following

TABLE.

1. $\cos x = \cos x;$
2. $2\cos^2 x = \cos 2x + 1;$
3. $4\cos^3 x = \cos 3x + 3\cos x;$
4. $8\cos^4 x = \cos 4x + 4\cos 2x + 3;$
5. $16\cos^5 x = \cos 5x + 5\cos 3x + 10\cos x;$
6. $32\cos^6 x = \cos 6x + 6\cos 4x + 15\cos 2x + 10;$
7. $64\cos^7 x = \cos 7x + 7\cos 5x + 21\cos 3x + 35\cos x;$
8. $128\cos^8 x = \cos 8x + 8\cos 6x + 28\cos 4x + 56\cos 2x + 35;$

and so on, to any extent that may be desired.

If m is an even number, and the members of

$$2\sin x\sqrt{-1} = y - \frac{1}{y}$$

are raised to the mth power, then, by proceeding as before, we shall clearly have

$$\pm\, 2^{m-1}\sin^m x = \cos mx - m\cos(m-2)x$$
$$+ \frac{m(m-1)}{1.2}\cos(m-4)x -, \&c.;$$

noticing, that $+$ must be used for $\pm$, in the first member of the equation, when m is exactly divisible by 4, and that $-$ must be used when it is divisible by 2, or not divisible by 4. It may be added, that there will here be $\frac{m}{2}$ terms containing cosines; together with the term

$$\pm\frac{1}{2}\,\frac{m(m-1)\times\ldots\times\left(\frac{m}{2}+1\right)}{1.2\times\ldots\ldots\ldots\times\frac{m}{2}},$$

in which + must be used for $\pm$ when m is divisible by 4; and when m is not divisible by 4, we must use —.

When m is an odd number, by proceeding as before, we shall have

$$\pm 2^{m-1} \sin^m x = \sin mx - m \sin (m-2) x$$
$$+ \frac{m(m-1)}{1 \,.\, 2} \sin (m-4) x -, \text{\&c.},$$

until the number of terms equals $\frac{m+1}{2}$; noticing, that + must be used for $\pm$ in the first member of the equation, when $m-1$ is divisible by 4; and that — must be used in the contrary case.

If 1, 2, 3, 4, 5, &c., are successively put for m in the preceding formulæ, we readily get the following

TABLE.

1. $\sin x = \sin x$;
2. $-2 \sin^2 x = \cos 2x - 1$;
3. $-4 \sin^3 x = \sin 3x - 3 \sin x$;
4. $8 \sin^4 x = \cos 4x - 4 \cos 2x + 3$;
5. $16 \sin^5 x = \sin 5x - 5 \sin 3x + 10 \sin x$;
6. $-32 \sin^6 x = \cos 6x - 6 \cos 4x + 15 \cos 2x - 10$;
7. $-64 \sin^7 x = \sin 7x - 7 \sin 5x + 21 \sin 3x - 35 \sin x$;
8. $128 \sin^8 x = \cos 8x - 8 \cos 6x + 28 \cos 4x + 56 \cos 2x + 35$;

and so on, to any required extent.

Resuming the simultaneous equations $2 \cos x = y + \frac{1}{y}$, and $2 \cos mx = y^m + \frac{1}{y^m}$, from p. 76; it is easy to perceive that they are equivalent to the equations

$$y^2 - 2y \cos x + 1 = 0, \quad \text{and} \quad y^{2m} - 2y^m \cos mx + 1 = 0.$$

Because these equations are coexistent, it is clear that the first is a quadratic factor of the second.

If we have an equation of the form

$$y^{2m} - 2y^m \cos\theta + 1 = y^{2m} - 2y^m \cos(\theta + 2n\pi) + 1 = 0,$$

since $\cos\theta = \cos(\theta + 2n\pi)$, n being an integer; then we shall have
$$y^2 - 2y\cos\left(\frac{\theta + 2n\pi}{m}\right) + 1,$$

for the general representative of its quadratic factors. Putting successively, 0, 1, 2, 3, &c., to $n = m - 1$ for n in the quadratic factor, we clearly get

$$y^{2m} - 2y^m \cos\theta + 1 = \left(y^2 - 2y\cos\frac{\theta}{m} + 1\right)$$

$$\times\left(y^2 - 2y\cos\frac{\theta + 2\pi}{m} + 1\right) \times \left(y^2 - 2y\cos\frac{\theta + 4\pi}{m} + 1\right), \text{ \&c.,}$$

to m factors. It is evident that these factors are different from each other, and that they are the only quadratic factors which the equation can have; since $n = m$, $n = m + 1$, $n = m + 2$, &c., will merely give repetitions of the factors found.

Thus, the quadratic factors of

$$y^6 - y^3 + 1 = y^6 - 2y^3 \times \frac{1}{2} + 1 = 0,$$

since $\cos\theta = \frac{1}{2}$ or $\theta = 60°$, will easily be found to be

$$y^2 - 1.8793852 . y + 1, \qquad y^2 - 1.5320888 . y + 1,$$

and $$y^2 - 0.3472964\ y + 1;$$

and in the same way, since

$$y^6 + y^3 + 1 = y^6 - 2y^3 \times -\frac{1}{2} + 1 = 0$$

gives $\cos\theta = -\frac{1}{2}$, we readily get $\theta = 120°$, and thence we shall have $y^2 - 1.5320888 . y + 1$, $y^2 - 0.3472964 . y + 1$, and $y^2 - 1.8793852 . y + 1$, for the quadratic factors.

If we have an equation of the form $y^{2m} - 2ay^m + 1 = 0$, in which a is numerically not greater than unity, it is clear that it may in like manner be resolved into quadratic factors. Consequently, if each quadratic factor is resolved into its two simple factors, the roots of the proposed equation will be known.

If $a = 1$, the equation becomes

$$y^{2m} - 2y^m + 1 = (y^m - 1)^2 = 0,$$

having

$$y^2 - 2y \cos\frac{2n\pi}{m} + 1$$

for its general quadratic factor, since $\cos 2n\pi = 1$. Putting 0, 1, 2, 3, ... to $m - 1$, inclusively for n, the particular quadratic factors will be found to be

$$y^2 - 2y + 1 = (y - 1)^2, \quad y^2 - 2y \cos\frac{2\pi}{m} + 1,$$

$$y^2 - 2y \cos\frac{4\pi}{m} + 1 \;.\;.\;.\;. \text{ to } y^2 - 2y \cos\frac{2(m-1)\pi}{m} + 1,$$

for the last factor. Because

$$\frac{2(m-1)\pi}{m} = 2\pi - \frac{2\pi}{m},$$

it is clear that

$$\cos\frac{(2m-1)}{m}\pi = \cos\frac{2\pi}{m},$$

and, in like manner,

$$\cos\frac{2(m-2)\pi}{m} = \cos\frac{4\pi}{m},$$

and so on; consequently, for

$$y^2 - 2y \cos\frac{2(m-1)\pi}{m} + 1,$$

we may write

$$y^2 - 2y \cos\frac{2\pi}{m} + 1;$$

for
$$y^2 - 2y\cos\frac{2(m-2)\pi}{m} + 1,$$

we may write
$$y^2 - 2y\cos\frac{4\pi}{m} + 1,$$
and so on.

Hence, we shall have

$$(y^m - 1)^2 = (y-1)^2 \cdot \left(y^2 - 2y\cos\frac{2\pi}{m} + 1\right)^2$$
$$\times\left(y^2 - 2y\cos\frac{4\pi}{m} + 1\right)^2, \text{ \&c.,}$$

to $\frac{m+2}{2}$ factors, when m is an even number; and to $\frac{m+1}{2}$ factors, when m is an odd number. Consequently, extract ing the square roots of these equal products, we shall have

$$y^m - 1 = (y-1)\cdot\left(y^2 - 2y\cos\frac{2\pi}{m} + 1\right)\cdot\left(y^2 - 2y\cos\frac{4\pi}{m} + 1\right)$$

&c., to $\frac{m+2}{m}$ factors when m is even, and to $\frac{m+1}{m}$ factors when m is an odd number.

Thus the factors of $y^6 - 1 = 0$, are

$$y-1,\ y^2 - 2y\cos\frac{2\pi}{6} + 1,\ y^2 - 2y\,.\cos\frac{4\pi}{6} + 1, \text{ and } y+1;$$

and those of $y^5 - 1 = 0$, are

$$y-1,\ y^2 - 2y\cos\frac{2\pi}{5} + 1, \text{ and } y^2 - 2y\cos\frac{4\pi}{5} + 1.$$

In like manner, if $a = -1$, our equation becomes

$$y^{2m} + 2y^m + 1 = (y^m + 1)^2 = 0;$$

whose general quadratic factor is

$$y^2 + 2y\,.\cos\frac{2n\pi + \pi}{m} + 1,$$

since
$$\cos(2n\pi + \pi) = -1.$$

Putting 0, 1, 2, 3, to $m-1$ inclusive, for n, and then proceeding as before, we get

$$y^2 - 2y\cos\frac{\pi}{m} + 1,\ y^2 - 2y\cos\frac{2\pi}{m} + 1,$$

$$y^2 - 2y\cos\frac{5\pi}{m} + 1, \text{ to } y^2 - 2y\cos\frac{2(m-1)\pi + \pi}{m} + 1.$$

Because

$$\frac{2(m-1)\pi + \pi}{m} = 2\pi - \frac{\pi}{m},\ \frac{2(m-2)\pi + \pi}{m} = 2\pi - \frac{2\pi}{m},$$

and so on; the factors may clearly be written in the forms

$$\left(y^2 - 2y\cos\frac{\pi}{m} + 1\right)^2, \left(y^2 - 2y\cos\frac{3\pi}{m} + 1\right)^2,$$

$$\left(y^2 - 2y\cos\frac{5\pi}{m} + 1\right)^2,$$

to $\frac{m}{2}$ factors when m is an even number and to $\frac{m+1}{2}$ factors when m is an odd number.

Hence, as before, the factors of $y^m + 1 = 0$, are expressed by $y^2 - 2y\cos\frac{\pi}{m} + 1$, $y^2 - 2y\cos\frac{3\pi}{m} + 1$, and so on, to $\frac{m}{2}$ or $\frac{m+1}{2}$ factors; accordingly, as m is an even or an odd number.

Thus, the factors of $y^6 + 1 = 0$, are

$$y^2 - 2y\cos\frac{\pi}{6} + 1,\ y^2 + 2y\cos\frac{3\pi}{6} + 1 = y^2 + 1,$$

and
$$y^2 - 2y\cos\frac{5\pi}{6} + 1;$$

while the factors of $y^5 + 1 = 0$, are

$$y^2 - 2y\cos\frac{\pi}{5} + 1,\ y^2 - 2y\cos\frac{3\pi}{5} + 1, \text{ and } y + 1.$$

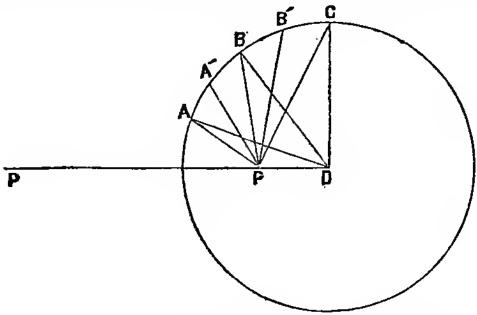
It results from what has been done, that any equation of the form $x^n \pm a^n = 0$, can be resolved into factors. For put $x^n = a^n y^n$, and the equation is readily reduced to the equivalent equations $y^n + 1 = 0$ and $y^n - 1 = 0$, whose roots can be found as before.

It is manifest that any equation of the form

$$x^{2m} - 2ax^m + b = 0,$$

can be reduced by the rules of quadratics to equations of the preceding forms, and their roots may be found, as before.

It may here be proper to notice some interesting properties of the circle, that result from what has been done



Thus, let AA′B, &c., be the circumference of a circle whose center is O, and radius R; then, supposing the circumference is divided into any number m of equal parts AB, BC, &c.; if from any point P, in the plane of the circle, the straight lines PA, PB, PC, &c., are drawn to the points of division of the circumference, we shall have the equation

$$\text{OP}^{2m} - 2\text{OP}^m \times \text{OA}^m \cos m\,(\text{AOP}) + \text{AO}^{2m}$$
$$= y^{2m} - 2y^m \cos\theta + 1;$$

where we represent the radius R = AO by 1, or unity, PO by y, and the angle POA by $\frac{\theta}{m}$. We also have from the triangles POA, POB, POC, &c.,

$$AP^2 = y^2 - 2y \cos \frac{\theta}{m} + 1,$$

$$BP^2 = y^2 - 2y \cos POB + 1 = y^2 - 2y \cos \frac{\theta + 2\pi}{m} + 1,$$

$$PC^2 = y^2 - 2y \cos \frac{\theta + 4\pi}{m} + 1, \text{ \&c.};$$

consequently, agreeably to De Moivre's *Property of the Circle*, we shall, from what is shown at p. 79, have

$$y^{2m} - 2y^m \cos \theta + 1 = PA^2 \times PB^2 \times PC^2 \times, \text{ \&c.},$$

to the square of the line drawn from P to the last point of division of the circumference.

If the angle POA = O, or A falls on OP, the preceding equation becomes

$$y^{2m} - 2y^m + 1 = (y^m - 1)^2 = PA^2 \times PB^2 \times, \text{ \&c.},$$

or $$\pm (y^m - 1) = PA \times PB \times PC \times, \text{ \&c.}$$

If the arcs AB, BC, &c., are each bisected in A′, B′, &c., then, since the lines drawn from P to all the points of division will be doubled in number, the preceding equation will become (for all the points of division of the circumference),

$$\pm (y^{2m} - 1^{2m}) = PA \times PA' \times PB \times PB' \times, \text{ \&c.},$$
$$= \pm (y^m - 1^m) \times PA' \times PB' \times, \text{ \&c.};$$

which gives

$$\frac{y^{2m} - 1^{2m}}{y^m - 1^m} = y^m + 1^m, \text{ or } y^m + 1 = PA' \times PB' \times PC' \times, \text{ \&c.}:$$

noticing, that the equations $\pm (y^m - 1) = PA \times PB \times PC \times$, &c., $y^m + 1 = PA' \times PB' \times PC' \times$, &c., are called Cotes's Properties of the Circle; see pp. 32 and 33 of Young's "Differential Calculus."

REMARKS.—There are one or two singular properties of circular functions that it may not be improper to notice in this connection.

Thus, resuming the equation $e^{x\sqrt{-1}} = \cos x + \sin x \sqrt{-1}$, from p. 53, and putting $x = \frac{\pi}{2}$, we have

$$e^{\frac{\pi}{2}\sqrt{-1}} = \sqrt{-1}, \quad \text{or} \quad e^{-\frac{\pi}{2}} = (\sqrt{-1})^{\sqrt{-1}};$$

which, expanded according to the ascending powers of $\frac{\pi}{2}$, by (b''), given at p. 51, gives

$$(\sqrt{-1})^{\sqrt{-1}} = 1 - \frac{\pi}{2} + \frac{1}{1.2}\left(\frac{\pi}{2}\right)^2 - \frac{1}{1.2.3}\left(\frac{\pi}{2}\right)^3 +, \text{\&c.},$$

for one of the properties.

And by taking the hyperbolic logarithms of the members of $e^{-\frac{\pi}{2}} = (\sqrt{-1})^{\sqrt{-1}}$, we have $-\frac{\pi}{2} = \sqrt{-1} \times \frac{1}{2} \log -1$, or $\pi = -\sqrt{-1} \log -1$, for the other property: noticing, that $\pi =$ the semicircumference of a circle whose radius $= 1$, and that e stands for the base of hyperbolic logarithms. See pp. 33 and 34 of Young's "Differential Calculus."

SECTION III.

VANISHING FRACTIONS.

(1.) WHEN the numerator and denominator of a fractional expression are each reduced to naught or vanish, by giving a particular value to a common variable, the expression is called *a vanishing fraction.*

Thus, $\frac{x^3 - a^3}{a(x^2 - a^2)}$ is a vanishing fraction: since, by putting a for x, it is reduced to $\frac{a^3 - a^3}{a(a^2 - a^2)} = \frac{0}{0}$. It is clear, from

$$\frac{x^3 - a^3}{a(x^2 - a^2)} = \frac{(x-a)(x^2 + xa + a^2)}{a(x-a)(x+a)},$$

that it is reduced to the form $\frac{0}{0}$, by putting a for x; since the factor $x - a$ (which is common to the numerator and denominator) becomes $a - a = 0$.

It is hence evident, *that vanishing fractions result from the vanishing of factors that are common to their numerators and denominators.*

(2.) Because the quotient arising from any division is manifestly independent of any factors that are common to the dividend and divisor, it is clear that by erasing such factors from the dividend and divisor (or dividing them by their greatest common divisor) before the particular value is put for the variable, and then putting the particular value in the result, we shall get the true value.

Thus, since $\dfrac{x^3 - a^3}{a(x^2 - a^2)} = \dfrac{(x-a)(x^2 + xa + a^2)}{a(x-a)(x+a)}$

is reduced to $\dfrac{x^2 + xa + a^2}{a(x+a)}$ by erasing the factor $x - a$ from its numerator and denominator; then, by putting a for x in $\dfrac{x^2 + xa + a^2}{a(x+a)}$, we get, after a slight reduction, $\frac{3}{2}$ for the true value of the proposed fraction, when a is put for x in it.

(3.) If for generality, we use $\dfrac{Fx}{F'x}$ to stand for any vanishing fractional form, which becomes $\frac{0}{0}$ when a is put for x; then, if A denotes the true value, we shall have $\frac{0}{0} = A$.

To find A, we may clearly put $\dfrac{Fx}{F'x} = A$, or $Fx = A \times F'x$; then to eliminate the vanishing factor, when it has neither a negative nor fractional exponent, we may differentiate the members of $Fx = A \times F'x$ on the supposition of the constancy of A, which will give $A = \dfrac{dFx}{dF'x}$; and if the right member of this for $x = a$ is reduced to $\frac{0}{0}$, we may evidently, as before, put $A = \dfrac{d^2Fx}{d^2F'x}$, and so on, until a fractional form will finally be obtained, in which both the numerator and denominator will not vanish when a is put for x; which will clearly be the true value of the proposed fraction.

Thus, to find the true value of the fraction $\dfrac{x^3 - 3x + 2}{3x^4 - 6x^2 + 3}$, when $x = 1$; which reduces it to the form $\frac{0}{0}$.

Here, Fx, F$'x$, and a, are represented by

$$x^3 - 3x + 2, \quad 3x^4 - 6x^2 + 3, \quad \text{and} \quad 1;$$

consequently, from

$$d\,(x^3 - 3x + 2) = (3x^2 - 3)\,dx$$

and $$d\,(3x^4 - 6x^2 + 3) = (12x^3 - 12x)\,dx,$$

we have $$A = \frac{3x^2 - 3}{12x^3 - 12x} = \frac{0}{0} \text{ when } x = 1.$$

Hence we have

$$\frac{d^2\,(x^3 - 3x + 2)}{d^2\,(3x^4 - 6x^2 + 3} = \frac{d\,(3x^2 - 3)}{d\,(12x^3 - 12x)} = \frac{6x}{36x^2 - 12};$$

which becomes $\frac{6}{36 - 12} = \frac{1}{4}$, when 1 is put for x, which is the true value of A, that of the proposed fraction, when 1 is put for x in it.

(4.) Still using $\frac{Fx}{F'x}$ to represent a fractional form that becomes $\frac{0}{0}$, when a is put for x; then, the vanishing factor that is common to the numerator and denominator, whatever may be its nature, can be eliminated from the fraction after the following manner:

Thus, put $a + h$ for x in Fx and F$'x$, and expand these functions by Taylor's Theorem, or in any other way, according to the ascending powers of h; and they (by omitting the vanishing terms) will evidently be reduced to the forms $Ah^a + Bh^b +$, &c., $A'h^{a'} + B'h^{b'} +$, &c. Hence, we shall have

$$\frac{Fx}{F'x} = \frac{F\,(a + h)}{F'(a + h)} = \frac{Ah^a + Bh^b +, \&c.}{A'h^{a'} + B'h^{b'} +, \&c.};$$

and hence it is clear that $\frac{A}{A'}\,h^{a-a'}$, when a is put for x, expresses the value of the proposed fraction. Thus, if a is greater than a', it is clear that the value of the fraction equals 0, since $\frac{A}{A'}h^{a-a'}=0$, when $h=0$; when $a=a'$, the value of the fraction is $\frac{A}{A'}$, since $a-a'=0$ reduces $h^{a-a'}$ to $h^0=1$; and when a' is greater than a, the value $\frac{A}{A'}h^{a-a'}=\frac{A}{A'h^{a'-a}}=$ infinity when $h=0$, on account of the infinitesimal divisor $h^{a'-a}$ in $\frac{A}{A'h^{a'-a}}$.

Hence, a fraction whose numerator and denominator are reduced to naught by a particular value (a) of the variable, may be found by the following

RULE.

1. Divide the differential or differential coefficient of the numerator, by the differential or differential coefficient of the denominator, and substitute the particular value of the variable in the result; then if the numerator and denominator of the fraction thus obtained are not both reduced to naught, it will be the value of the vanishing fraction.

If, however, the numerator and denominator of this fraction vanish; then we must proceed with the second differentials or differential coefficients of the numerator and denominator in the same way as before; and so on, until a fraction is obtained whose numerator and denominator do not both vanish for the particular value of the variable; which will, of course, be the correct value of the vanishing fraction.

2. If in the preceding process any differential coefficient becomes infinite, for the particular value (a) of the variable; then, we must, as at p. 88, change the variable into $a+h$, in the numerator and denominator of the proposed fraction, and

develop, by particular processes, the numerator and denominator into the forms $Ah^a + Bh^b +$, &c., and $A'h^{a'} + B'h^{b'} +$, &c., arranged according to the ascending powers of h; then, as at p. 88, the true value of the vanishing fraction will be expressed by $\frac{A}{A'}h^{a-a'}$, when $h = 0$; which equals 0, $\frac{A}{A'}$, or infinity, accordingly as $a - a'$ is positive, naught, or negative.

REMARK.—Examples that do not fall immediately under this rule can often be reduced to it, and thence their values found.

EXAMPLES.

1. The value of $\frac{a^2 - a^2x - x^2 + x^3}{b^2 - b^2x - x^2 + x^3}$, when 1 is put for x, is $\frac{a^2 - a^2}{b^2 - b^2}$; and the value of $\frac{3a^2 - 2ax - x^2}{5x - 5a}$, when $x = a$, is $-\frac{4a}{5}$.

2. To find the value of $\frac{\sqrt{a^2 - 3ax + 2x^2}}{\sqrt{x^3 - a^3}}$, when $x = a$. Put $a + h$ for x, and the expression is immediately reducible to $\sqrt{\frac{a + 2h}{3a^2 + 3ah + h^2}}$; which, by putting $h = 0$, gives $\sqrt{\frac{1}{3a}}$ for the answer.

Otherwise. Representing the sought value by A, we easily get $\quad a^2 - 3ax + 2x^2 = A^2(x^3 - a^3)$,

which gives $\quad 2x - a = A^2(x^2 + xa + x^2)$,

by erasing the factor $x - a$ from its members; consequently, putting a for x, the answer is $A = \frac{1}{\sqrt{3a}}$.

3. To find $\frac{\sqrt{2x^2 - x - 1}}{\sqrt{x^3 - 1}}$, when 1 is put for x.

Putting $1 + h$ for x, the expression reducesto

$$\frac{(3h + 2h^2)^{\frac{1}{2}}}{(3h + 3h^2 + h^3)^{\frac{1}{3}}} = \frac{3h + 2h^2)^{\frac{3}{6}}}{(3h + 3h^2 + h^3)^{\frac{2}{6}}} = \left(\frac{27h^3 +, \&c.}{9h^2 \ +, \&c.}\right)^{\frac{1}{6}} =$$

$$(3h \pm, \&c.)^{\frac{1}{6}};$$

which, by putting $h = 0$, gives naught for the true value of the proposed fraction, when 1 is put for x.

4. To find the value of $\frac{1}{a^2 - x^2} \div \frac{1}{a - x}$, when $x = a$.

When $x = a$, the dividend and divisor are evidently unlimitedly great, instead of being infinitesimals, as in the preceding examples.

Performing the division before putting a for x, we get

$$\frac{1}{a^2 - x^2} \div \frac{1}{a - x} = \frac{1}{a + x};$$

consequently, putting a for x, the answer is $\frac{1}{2a}$.

5. To find the value of the difference $\frac{x}{x - 1} - \frac{1}{\log x}$, when $x = 1$; the logarithm being hyperbolic.

Reducing the terms of the proposed expression to a common denominator, gives the fraction $\frac{x \log x - (x - 1)}{(x - 1) \log x}$; which is under the form of a vanishing fraction.

Dividing the second differential coefficient of the numerator of this fraction by that of its denominator gives $\frac{\frac{1}{x}}{\frac{1}{x} + \frac{1}{x^2}}$ for the quotient; which, by putting 1 for x, gives $\frac{1}{2}$ for the answer.

6. To find the value of the product $(x-1)\tan\frac{\pi}{2}x$, when 1 is put for x.

When $x-1=0$, $\tan\frac{\pi}{2}x$ becomes $\tan\frac{\pi}{2}=$ infinity; consequently, one of the factors equals 0, while the other is infinite.

Since $\tan\frac{\pi}{2}x = \dfrac{1}{\cot\frac{\pi x}{2}}$, the product becomes $\dfrac{x-1}{\cot\frac{\pi x}{2}}$, which is a vanishing fraction; since its numerator and denominator both vanish when $x=1$.

Consequently, dividing the differential coefficient of the numerator of this fraction by that of its denominator, we get $-\dfrac{2\times\sin^2\frac{\pi}{2}x}{\pi}$, which, by putting 1 for x, since $\sin\frac{\pi}{2}=1$, gives $-\frac{2}{\pi}$ for the answer; and in much the same way, the value of $\dfrac{\tan\frac{\pi}{2}x}{x}$, when $x=0$, is infinite.

7. To find the value of $\dfrac{e^x - e^{\sin x}}{x-\sin x}$, when $x=0$.

From (b'') page 51, we have

$e^x = 1 + x + \frac{x^2}{1.2}+$, &c., and $e^{\sin x}=1+\sin x+\frac{\sin^2 x}{1.2}+$, &c.; consequently,

$$(e^x - e^{\sin x}) \div (x-\sin x) = 1 + \frac{x+\sin x}{1.2}+, \text{\&c.},$$

which, by putting $x=0$, gives 1 for the answer.

8. The values of $\dfrac{2x-\sin x}{x}$ and $\dfrac{x^x - x}{1-x}$, when $x=0$ and 1, are 1 and 0.

9. To find the value of $\frac{\sqrt{24x^{10}+a^{10}}-5a^4x}{x^4-a^4}$, when a is put for x.

Put $a+h$ for x, and the answer will be found to be $\frac{19a}{4}$, or $5a$ more nearly.

10. The values of $\frac{x^{-1}-x^{-2}}{x-1}$ and $\frac{ax^{-3}-a^{-2}}{ax^{-2}-a^{-1}}$, when 1 and a are put for x, are 1 and $\frac{3}{2a}$.

11. The value of $\frac{ax-x^2}{a^4-2a^3x+2ax^3-x^4}$, when $x=a$, is unlimitedly great; and that of

$$\frac{(x^2-a^2)^{\frac{3}{2}}}{(x-a)^{\frac{3}{2}}}=\left(\frac{x^2-a^2}{x-a}\right)^{\frac{3}{2}}=(x+a)^{\frac{3}{2}},$$

when $x=a$, is $(2a)^{\frac{3}{2}}$.

12. The values of

$$\frac{x^{n+1}-a^{n+1}}{x^n-a^n} \quad \text{and} \quad \frac{x^2-5ax+4a^2}{3x^2-7ax+4a^2},$$

when a is put for x, are $\frac{n+1}{n}a$, and 3.

13. The value of $\frac{\tan x-\sin x}{x^3}$, when $x=0$, is $\frac{1}{2}$.

For most of the preceding examples the reader may be referred to pages 60 and 61 of Young's "Differential Calculus."

SECTION IV.

MAXIMA AND MINIMA.

(1.) A VALUE of a function greater than the immediately preceding and following values is called a *maximum*, while a value less than those values is called a *minimum*.

Thus, since three successive values of a function of any variable, as x, may clearly be expressed by the forms $F(x-h)$, Fx, and $F(x+h)$; Fx will be a maximum or mininum, accordingly as it is greater or less than each of the other values, from any finite value of h (however small), to $h=0$.

(2.) Hence, supposing the functions $F(x-h)$ and $F(x+h)$ to be converted into series arranged according to the ascending powers of h, they may clearly be expressed by the forms

$$Fx + A(-h)^a + B(-h)^b +, \&c.,$$

and $$Fx + A(h)^a + B(h)^b +, \&c.,$$

in which A, B, &c., are supposed to be independent of h, while the index a is less than b, b less than c, and so on; these series (like the functions they represent) being each less or greater than Fx from a very small value of h to $h=0$. It is clear that these expansions may be written in the forms

$$F(x-h) = Fx + (-h)^a [A + B(-h)^{b-a} + C(-h)^{c-a} +, \&c.],$$

and $$F(x+h) = Fx + h^a [A + Bh^{b-a} + Ch^{c-a} +, \&c.];$$

in which the indices $b-a$, $c-a$, &c., are clearly all positive.

If A is different from 0, it is clear that so small a finite value may be given to h, that A shall be greater than the sum of all the other terms within the braces, in the expansions; consequently, when Fx is a maximum or minimum, the terms $A(-h)^a$ and Ah^a must accordingly, each be negative or positive. Hence, a must evidently be either an even integer, or a vulgar fraction which (in its lowest terms) has an even integer for numerator and an odd integer for its denominator; and A must be negative or positive, accordingly as Fx is a maximum or minimum.

(3.) Regarding x and h as indeterminates, we may, by Taylor's Theorem for the above formulas, write

$$F(x-h) = Fx - \frac{d(Fx)}{dx}h + \frac{d^2(Fx)}{dx^2}\frac{h^2}{1.2} - \frac{d^3(Fx)}{dx^3}\frac{h^3}{1.2.3} +, \&c.,$$

and

$$F(x+h) = Fx + \frac{d(Fx)}{dx}h + \frac{d^2(Fx)}{dx^2}\frac{h^2}{1.2} + \frac{d^3(Fx)}{dx^3}\frac{h^3}{1.2.3} +, \&c.$$

To reduce these expansions to the preceding conditions, we must put the coefficient of h equal to naught, or assume the equation $\frac{d(Fx)}{dx} = 0$; and the expansions will be reduced to $F(x-h = Fx + \frac{d^2(Fx)}{dx^2}\frac{h^2}{1.2} - \frac{d^3(Fx)}{dx^3}\frac{h^3}{1.2.3} +, \&c.,$

and $F(x+h) = Fx + \frac{d^2Fx}{dx^2}\frac{h^2}{1.2} + d^3(Fx)\frac{h^3}{1.2.3} +, \&c.,$

which are clearly of the requisite forms, since h^2 is the lowest power of h, in them.

When a function is a maximum or minimum, any constant factor or divisor of it may be omitted, and *vice versa.* Also, any positive power or root of a maximum or minimum, must also be a maximum or minimum. And the reciprocal of a maximum is a minimum; and that of a minimum is a maximum.

(4.) It is manifest that the maxima and minima of a function of a single variable may be found by the following

RULE.

1. To find when y, a function of x, is a maximum or minimum; put the first differential coefficient $\frac{dy}{dx} = 0$, and find the real roots of the equation. Substitute each real root in $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$, &c., until the first which does not vanish is obtained; then, if it is of an odd degree, it can not correspond to a maximum or minimum of y; while if it is of an even degree, it will correspond to a maximum or minimum of y, accordingly as its sign is negative or positive.

2. To find other maxima, and maxima that may result from the unlimited increase of $\frac{dy}{dx}$, we put $\frac{dy}{dx} =$ infinity; or, which comes to the same, we assume its reciprocal $\frac{dx}{dy} = 0$; and find the real roots of this equation. Then, the roots which, put for x in y, make it greater than its adjacent values, will give maxima; while those which make y less than its adjacent values, give minima: noticing, that those roots which do not make y a maximum or minimum, can not correspond to the maxima and minima of the question.

3. If any real root of $\frac{dy}{dx} = 0$, when substituted as directed in 1, makes the first differential coefficient, which does not vanish, infinite, then the true value of the term must

be found by the ordinary processes of algebra, and thence the corresponding maximum or minimum may be determined.

EXAMPLES.

1. To find the maximum and minimum of

$$y = 2x^3 - 9x^2 + 12x - 7.$$

Here $\frac{dy}{dx} = 0$ becomes $x^2 - 3x + 2 = 0$; whose roots are $x = 1$ and $x = 2$. Substituting $x = 1$ in $\frac{d^2x}{dx^2} = 2x - 3$, it becomes -1, which being negative shows that if we put 1 for x in y, we shall get its maximum. Also, putting 2 for x in $\frac{d^2y}{dx^2} = 2x - 3$, it becomes $\frac{d^2y}{dx^2} = 1$; which, being positive, shows if we put 2 for x in y, we shall get its minimum value.

2. To find the minimum value of $y = x^2 - (a + b)x + ab$.

Here $\frac{dy}{dx} = 0$ becomes $2x - (a + b) = 0$, which gives $x = \frac{a+b}{2}$, and $\frac{d^2y}{dx^2} = 2$; consequently, putting $\frac{a+b}{2}$ for x in y, we have $y = -\left(\frac{a-b}{2}\right)^2$, a minimum.

3. The minima values of

$$y = x^2 - 2ax + a^2 + b = (x - a)^2 + b, \text{ and } y = (x - a)^4,$$

are evidently $y = b$, $y = 0$; while $y = (x - a)^3$, admits of neither a maximum nor minimum.

4. To divide $2m$ into two parts, whose product shall be a maximum.

Because $m + x$ and $m - x$ when added equal $2m$, they may clearly stand for the parts; consequently, the product of the parts is expressed by $(m + x)\ (m - x) = m^2 - x^2$, which is clearly a maximum when $x = 0$, which shows that the parts are equal.

REMARK.—It is hence easy to perceive that the number nm, when divided into n equal parts, gives m^n for their maximum product.

5. To find the maxima and minima of $y = a \pm (x - b)^{\frac{2}{3}}$.

Here, we have $\frac{dy}{dx} = \pm \frac{2}{3} (x - b)^{-\frac{1}{3}} = \pm \frac{2}{3} \frac{1}{(x - b)^{\frac{1}{3}}}$; which shows that $x = b$ makes $\frac{dy}{dx}$ unlimitedly great, or reduces $\frac{dx}{dy} = \frac{2}{3} (x - b)^{\frac{1}{3}}$ to naught, agreeably to 2 of the rule.

By putting $x = b - h$, we evidently have $y = a \pm h^{\frac{2}{3}}$; which by (2.), p. 94, makes $y = a$, a maximum when $-$ is used for $\pm$, and the reverse.

6. To find the maximum and minimum of $y = a \pm (x - b)^{\frac{4}{3}}$.

From $\frac{dy}{dx} = \pm \frac{4}{3} (x - b)^{\frac{1}{3}}$ and $\frac{d^2y}{dx^2} = \pm \frac{4}{9} (x - b)^{-\frac{2}{3}}$, we get, by putting $\frac{dy}{dx} = 0$, $x = b$, which makes $\frac{d^2y}{dx^2} =$ infinity.

Hence, by 3 of the rule, put $x = b + h$, and we get $y = a \pm (h)^{\frac{4}{3}}$; which shows that by using -- for $\pm$, $x = b$ makes $y = a$, a maximum, and the reverse.

7. Given $y = \sqrt{a^3x - ax^3}$ to find when y is a maximum or minimum.

Here we easily get $\frac{y^2}{a} = a^2x - x^3$, for which we may evidently take $u = a^2x - x^3$, and, agreeably to the remarks at the bottom of p. 95, find the maximum and minimum of u.

From $\frac{du}{dx} = 0$, and $\frac{d^2u}{dx^2} = -6x$, we get $x = \frac{a}{\sqrt{3}}$ and

$x = -\frac{a}{\sqrt{3}}$; and by putting $\frac{a}{\sqrt{3}}$ for x in $\frac{d^2u}{dx^2}$, we have a negative result; which shows that $x = \frac{a}{\sqrt{3}}$ makes y a maximum: noticing, that $x = -\frac{a}{\sqrt{3}}$ makes y imaginary.

8. To solve the equations $x + y + z = a$, $x^2 + y^2 = b^2$, and $xy^2 =$ a maximum or minimum; or to find x, y, and z, from the equations and the maximum or minimum condition.

Putting, according to the second and third conditions, their differentials equal to naught, we evidently have

$$xdx + ydy = 0, \quad \text{and} \quad 2xydy + y^2dx = 0;$$

consequently, since the first of these gives $ydy = -xdx$, the second, by substitution, becomes $(y^2 - 2x^2)dx = 0$, or $y^2 = 2x^2$.

Hence, the second of the proposed equations, by putting $2x^2$ for y^2, is reduced to $3x^2 = b^2$; whose solution gives

$$x = \frac{b}{\sqrt{3}} \quad \text{and} \quad x = -\frac{b}{\sqrt{3}};$$

noticing, that $x = \frac{b}{\sqrt{3}}$ makes xy^2 a maximum, and $x = \frac{-b}{\sqrt{3}}$ makes it a minimum.

Having found x, we easily get y from $x^2 + y^2 = b^2$, and thence z will be found from $x + y + z = a$.

Remarks.—1. It is hence easy to perceive that we may proceed in much the same way to solve all questions of an analogous nature.

2. The preceding solution may be modified as follows: From the second equation we have $y^2 = b^2 - x^2$, which reduces the maximum or minimum condition to the maximum or minimum of $b^2x - x^3$; consequently, representing this by u,

we have to find x such that $u = b^2x - x^3$ shall be a maximum or minimum.

Hence, from $\frac{du}{dx} = 0$, or $b^2 - 3x^2 = 0$, and $\frac{d^2u}{dx^2} = -6x$, we get the same results as before.

9. Given $x + y + z = a$, and $x^m y^n z^p$,

or $m \log x + n \log y + p \log z$,

a maximum, to find x, y, and z.

By taking the differentials, we have $dx + dy + dz = 0$,

or $dz = -dx - dy$, and $\frac{mdx}{x} + \frac{ndy}{y} + \frac{pdz}{z} = 0$;

consequently, substituting the value of dz, we have

$$\frac{mdx}{x} - \frac{pdx}{z} + \frac{ndy}{y} - \frac{pdy}{z};$$

which, on account of the arbitrariness of dx and dy, is clearly equivalent to the equations

$$\frac{m}{x} - \frac{p}{z} = 0, \quad \text{or} \quad \frac{m}{p} = \frac{x}{z} \quad \text{and} \quad \frac{n}{p} = \frac{y}{z}.$$

Hence, to the sum of these equations adding the identical equation $\frac{p}{p} = \frac{z}{z}$, we have

$$\frac{m+n+p}{p} = \frac{x+y+z}{z} = \frac{a}{z} \quad \text{or} \quad z = \frac{ap}{m+n+p}$$

and thence from $x = \frac{mz}{p}$ and $y = \frac{nz}{p}$, we readily get

$$x = \frac{am}{m+n+p} \quad \text{and} \quad y = \frac{an}{m+n+p}$$

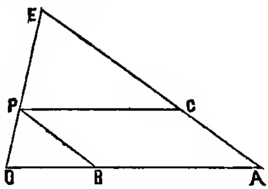
To perceive that the preceding results satisfy the required conditions, the reader may consult Lacroix, "Calcul Dif.," vol. i., p. 380.

10. To find x, such that $\frac{x}{x^2 + c^2}$ shall be a maximum.

According to what is stated at p. 95, the question will be solved by making the reciprocal of the proposed expression, or $\frac{x^2 + c^2}{x} = x + \frac{c^2}{x}$, a minimum.

Because $x \times \frac{c^2}{x} = c^2$, it is clear that the minimum of $x + \frac{c^2}{x}$ is found by putting $x = c$; which gives $x : c :: c : \frac{c^2}{x}$; consequently [see my Algebra (24.), at the top of p. 197], we must have $x + \frac{c^2}{x}$ greater than $c + c = 2c$, if x and $\frac{c^2}{x}$ are unequal.

11. Given the angle A and the position of the point P between the lines that form it, to draw the right line DE through P such that the triangle ADE shall be a minimum.



Through P draw right lines parallel to the lines that form the given angle and meeting them in B and C; then, DPE being drawn to cut off the minimum triangle, the triangles PBD and PCE are evidently equiangular and of course similar, from well-known principles of geometry; and the area of the parallelogram PBAC is evidently given, from the data of the question.

Representing PB $=$ AC, PC $=$ AB, BD, and CE, by the letters a, b, x, and y, we get from the similar triangles the proportion $x : a :: b : y = \frac{ab}{x}$; consequently,

$$AD = x + b \text{ and } AE = a + y = a + \frac{ab}{x} = \frac{a\,(x+b)}{x},$$

and thence we have the area of the triangle ADE expressed by

$$\frac{AD \times AE \times \sin A}{2} = \frac{a\,(x+b)^2}{2x} = \frac{a}{2}\left(x + 2b + \frac{b^2}{x}\right) \times \sin A,$$

as is evident from the well-known expression for the area of a triangle in terms of any two of its sides and their included angle.

From the preceding expression, since sin A, $\frac{a}{2}$, and b are given, it clearly follows (from principles heretofore given), that the triangle will be a minimum when $x + \frac{b^2}{x}$ is a minimum.

Hence, from the solution of the preceding example, we must have $x = b$, or BD = BA, from which it clearly follows that making BD = BA = b, and drawing DPE, DEA will be the required triangle; and P bisects DE.

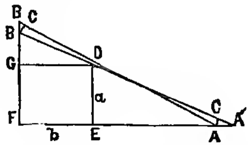
12. "Given the sum of the base and curve surface of a right cylinder, to find when its solidity is a maximum."

Let r and h stand for the radius of the base and height of the cylinder, and $\pi = 3.14159$, &c. = the semicircumference of a circle whose radius = 1; then, if A stands for the sum of the base and curve surface, we shall, from the known principles of mensuration, get $2r\pi h + r^2\pi = A$ and $r^2\pi h = s =$ the solidity of the cylinder = a maximum. From these conditions, we readily get $2s = Ar - r^3\pi =$ a maximum, which gives $\frac{2ds}{dr} = 0$ or $A - 3r^2\pi = 0$ or $r = \sqrt{\frac{A}{3\pi}}$.

From the addition of $2r\pi h + r^2\pi - A$ and $A - 3r^2\pi = 0$, we get $h = r$, or the height of the cylinder equals the radius of its base.

Remark.—In much the same way, if the whole surface is given, when the cylinder is a maximum we shall have $r = \sqrt{\frac{A}{6\pi}}$, and $h = 2r$, by using A to represent the whole surface.

13. Find the longest straight pole that can be put up a chimney, when the height from the floor to the mantel $= a$, and the depth from front to back $= b$



Let D represent the mantel, and AB the pole passing through it, meeting the floor in A, and the back of the chimney in B; then $DE = a$ and $DG = EF = b$.

Representing AE by x, the right-angled triangle ADE gives $AD = \sqrt{(a^2 + x^2)}$, and then from the similar triangles ADE and ABF we have the proportion

$$AE : AD :: AF : AB = \frac{AD}{AE} . AF = \frac{\sqrt{(a^2 + x^2)}}{x} (b + x) =$$

the length of the pole $=$ a maximum; consequently, $(a^2 + x^2)\left(\frac{b}{x} + 1\right)^2$ must be a maximum. Putting the differential of this equal to naught, we readily get the equation

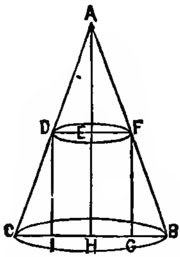
$$x + b - \frac{b}{x^2} (a^2 + x^2) = 0,$$

which gives $x = \sqrt[3]{a^2b}$, as required.

Otherwise. Supposing AB to be the position of the rod, let it be slightly changed into the position A′B′, by revolving about D; then (ultimately), its change A′C′ at the extremity A must equal its change at the extremity B, and have a contrary sign; consequently, the approximate position of the rod can be easily found by trial.

It clearly follows from what has been done, that we shall have AD : DB :: tan ang A : tan ang B, or $x : b : \frac{a}{x} : \frac{x}{a}$, or $\frac{x^2}{a} = \frac{ab}{x}$, which gives $x = \sqrt[3]{a^2b}$, the same as before.

14. To find when the cylinder DIGF inscribed in the cone ABC is a maximum.



Represent the base and height of the cone by A and a, and the height of the cylinder by x, then $a - x$ represents the height AE of the cone whose base is DF the upper base of the cylinder. From well-known principles of geometry, we have

$$AH^2 : AE^2 :: \text{base BC} : \text{base DF} =$$
$$\frac{AE^2}{AH^2} \times \text{base BC} = \frac{A}{a^2} \times (a - x)^2;$$

consequently, multiplying this by x, the height of the cylinder, we have $\frac{A}{a^2}(a - x)^2 x$ for its contents.

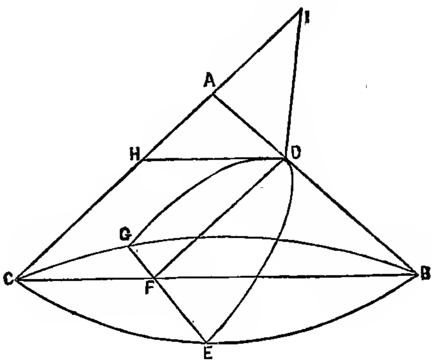
Hence, because $\frac{A}{a^2}$ and a are invariable $(a - x)^2\, x$ must be a maximum, whose differential, put equal to naught, gives

$$-2x dx\,(a - x) + (a - x)^2\, dx = 0 \text{ or } -2x + a - x = 0.$$

This solved, gives $x = \frac{a}{3}$, or *the height of the cylinder is one-third of that of the cone.*

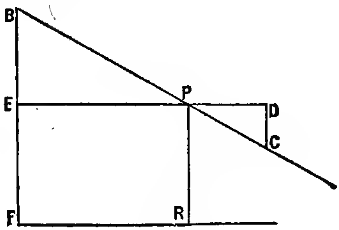
REMARK.—It may be shown, in much the same way, that the height of the maximum rectangle in any triangle *is half the height of the triangle.*

15. "To cut the greatest parabola from a given cone."



Let ABC be a triangular section of the cone by a plane passing through its axis at right angles to its base, and suppose that the sought parabola passes through F in BC, then, drawing the lines GE and FD through F, parallel to the tangent to the circumference of the base at C and to the side of the cone AC, meeting the circumference of the base in the points G and E and the side AB of the cone in D, the curvilinear section GDE of the conical surface and the plane of the lines GE and FD will, according to the common definition, be a parabola; having DF for its axis and D for its principal vertex, FE and FG, which are evidently equal and perpendicular to BC, being called ordinates to the axis. If through D, DH is drawn parallel to FC, and DI drawn above DH so as to make the angle HDI equal to the angle A or FDB; then HI, the part of the side of the cone between the lines HD and ID, will be what is called *the principal parameter* or *latus rectum* of the parabola, it being the parameter or latus rectum of its axis. Since the angles D and H of the triangle HDI are equal to the angles D and F of the triangle FDB, these triangles are clearly equiangular and give the proportion HI : DH or FC :: FB : FD, or its equivalent, $HI \times DF = CF \times FB = FE^2$, by a well-known property of the circle. Representing HI by p, DF by x, and FE by y, the preceding equation becomes $px = y^2$, the well-known equation of the parabola; which, by knowing p and assuming x, will enable us to find the corresponding values, $+y$ and $-y$, of y, so that the curve may clearly be constructed by points, according to the common methods. Because the area of the parabola GDE equals $\frac{2}{3}$ GE × DF $= \frac{4}{3}xy$, it is clear, since the area is a maximum, that xy must also be a maximum. If we represent the diameter of the base BC by a and BF by z, we shall get $CF = a - z$; which give $y^2 = az - z^2$, from a well-known property of the circle. Because the angles of the triangle BDF do not change for different positions of the parabola, it is clear that DF will vary as BF or z; consequently, xy may be represented by $z\sqrt{(az - z^2)}$ and $az^3 - z^4$ must be a maximum. By putting the differential of this equal to naught, we have $3az^2 - 4z^3 = 0$, which gives $z = \frac{3a}{4}$, which, of course, gives the position of the parabolic section, when it is a maximum.

16. "To find the position of a straight rod or beam, when it rests in equilibrio on a prop in a vertical plane, having one of its ends in contact with a vertical wall, which is at right angles to the vertical plane of the rod."



Let BC be half the beam (supposed of uniform density and dimensions) on the prop PR, and having its end B in contact with the vertical wall EF, whose plane cuts the vertical plane of the rod perpendicularly; then, through P draw DE perpendicular to the plane of the wall, and DC through C, the center of gravity of the beam, perpendicular to the direction of EP, meeting its production in D; then, since the beam is in equilibrio, it results from well-known *principles of mechanics* that DC must be a maximum. Put BC = half the length of the beam $= b$, and PE the distance of the prop from the wall $= a$, and represent the angle BPE = CPD by ϕ; and we shall have

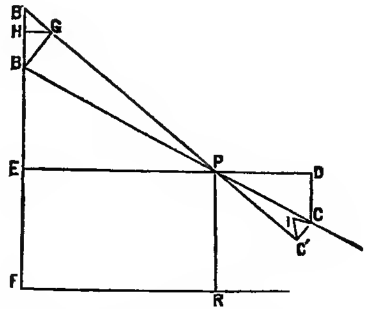
$$BC \sin\phi = b \sin\phi = BE + CD,$$

also we have $\qquad BE = PE \times \tan\phi = a \tan\phi,$

and hence, by subtraction, we have

$$b \sin\phi - a \tan\phi = DC = \text{a maximum}.$$

Hence, putting the differential of this equal to naught, we have $b\cos\phi - a \div \cos^2\phi = 0$, which gives $\cos\phi = \sqrt[3]{\frac{a}{b}}$, as required.



Another solution.—Let the figure be constructed, and the same notation used as before: then let the beam be slightly changed in position, first, by giving it a slight angular motion, in its vertical plane, about P until it takes the position C′G; second, by sliding it along P, without any angular motion, until G coincides with B′ in the vertical wall.

From P as a center, with radii PC and PB describe the arcs CC′ and BG, cutting BC and B′C′ in the points C and C′, B and G, and suppose the horizontal line through C meets a vertical line through C′ in I; then IC′ clearly represents the vertical motion of the point C (or the center of gravity of the beam), resulting from the angular motion.

Because the arcs CC′ and BG, on account of their (supposed) minuteness, may be regarded as right lines, which cut BC and B′C′ perpendicularly on account of the minuteness of the angular motion $d\phi$, it is easy to perceive that the triangles CIC′ and CDP will (ultimately) be similar.

It is also easy to perceive, if the perpendicular to the wall through G meets it in H, that the triangles BPE and GBH will (ultimately) be similar to each other, and to the triangles

DPC and C′IC; hence, we have $b \times d\phi = CC' + BG$ and $C'I = CC' \cos \phi$, $BH = BG \cos \phi$, and of course $C'I + BH = b \cos \phi d\phi$.

If $B'H = C'I$, it is easy to perceive that the center of gravity of the beam will be raised by sliding it along P (or keeping its end in contact with the vertical wall), through C′I; consequently, since the center of gravity neither ascends nor descends, the beam must clearly be in equilibrio, as required. Now from the triangle BPE, we have

$$BP = a \times \sec \phi = \frac{a}{\cos \phi}:$$

and thence from BPG we get $BG = \dfrac{ad\phi}{\cos \phi}$; consequently, since ang BGB′ differs insensibly from a right angle, we have

$$BG \sec \phi = \frac{ad\phi}{\cos^2 \phi} = BB'.$$

Hence, when the beam is in equilibrio, we must from $BB' = C'I + BH = b \cos \phi d\phi$ have $b \cos \phi d\phi = \dfrac{ad\phi}{\cos^2 \phi}$; or, from a slight reduction $\cos \phi = \sqrt[3]{\dfrac{a}{b}}$, the same result as found from the preceding solution.

Remark.—Thus, it clearly follows that the question admits of a most elegant solution, which does not require the use of any principles that depend on maxima or minima.

17. "Given two elastic bodies A and C, to find an intermediate body x, such that if A strikes it with a given velocity a, the motion communicated through x to C may be a maximum."

Because Aa is the momentum of A before impact, it results, from the laws of collision of elastic bodies, that $\frac{2A \times a}{A + x}$ is the velocity of x after impact; and in the same way x communicates the velocity $\frac{4Aax}{(A+x)(x+C)}$ to C, which must be a maximum; consequently, its reciprocal

$$\frac{(A+x)(x+C)}{4Aax} = \frac{A+C}{4Aa} + \left(x + \frac{AC}{x}\right) \div 4Aa$$

must be a minimum, which clearly requires $x + \frac{AC}{x}$ to be a minimum. By putting the differential of this equal to naught, we get $1 - \frac{AC}{x^2} = 0$; which gives $x = \sqrt{AC}$, the same result that the process explained on page 101 will give.

REMARKS.—1. Putting $\frac{x}{A} = \frac{\sqrt{AC}}{A} = \sqrt{\frac{C}{A}} = m =$ the ratio of x to A; we readily get the geometrical progression A, mA, m^2A, m^3A, &c.; which may be supposed to result, as in the question, from the communication of motion from A through mA to m^2A; and from mA through m^2A to m^3A; and so on, to any extent.

2. It is also clear, that a, $\frac{2a}{1+m}$, $\frac{4a}{(1+m)^2}$, $\frac{8a}{(1+m)^3}$, &c., are the velocities of the successive bodies, which are also in geometrical progression; and that

$$Aa,\quad \frac{2m}{1+m}Aa,\quad \left(\frac{2m}{1+m}\right)^2 Aa,\quad \left(\frac{2m}{1+m}\right)^3 Aa,\ \&c.,$$

also in geometrical progression, severally express the momenta of the bodies. Hence, if $m = 1$, or if the bodies equal each other, they will have equal momenta; while, if the bodies are unequal, their momenta will increase or decrease, accordingly as m is greater or less than unity.

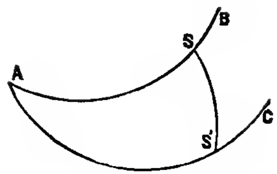
3. As a result of these principles, it has been proposed to construct speaking and hearing trumpets that shall be more effective than those previously used.

Thus, let the trumpet be a tube, such that its section at right angles to its length shall always be a circle, and that, when the distances of the sections from the place of the mouth or ear increase in arithmetical progression, the radii of the sections shall increase in geometrical progression; then, if x represents the distance of any section from the place of the mouth or ear and y the radius of the section, we may express the connection between x and y by the equation $\log y = x$, or, by taking a for the base of the logarithms, we shall have $y = a^x$, the well-known equation of the logarithmic curve; whose revolution around its axis or the axis of x, which is an asymptote to the curve, will generate the proper figure or form of the trumpet.

4. Now, the trumpets being filled with the air, which is a very elastic substance, when they are used either in speaking or hearing, it is clear that the sections of the tubes, regarded as of very small equal thicknesses, may be supposed to communicate motion successively to each other, like the elastic bodies described above; so that there will be an increase of momentum from the less to the greater sections in the speaking trumpet, and a decrease of momentum in proceeding from the greater to the less sections in the hearing trumpet.

5. It is easy to perceive that there will be equal momenta communicated from section to section, through a prismatic, or cylindric column of air; noticing, that the sections are always supposed to be perpendicular to the lengths of the columns.

18. Supposing the ecliptic to be a circle, it is required to find when the equation of time is a maximum.



Let AB and AC stand for the ecliptic and equator, regarded as great circles of the celestial sphere, A the first point of Aries and S the place of the sun when the equation of time is a maximum; then, the arc of a great circle of the (celestial) sphere drawn from the sun perpendicular to the equator meeting it in S′, gives AS, AS′, and SS′ for the sun's longitude, right ascension, and declination (supposed north, for simplicity); now, since time is reckoned by the sun's motion from west to east or in the direction of the equator, or, which comes to the same, because S and S′ are on the same celestial meridian, it is clear that the time shown by the sun at S must be the same as if it was placed at S′; whereas, if the angle $A = 0$, or the ecliptic coincides with the equator, the point S will be reduced to the equator so that AS will represent the sun's right ascension; and of course AS − AS′, when reduced to time at the rate of 15° to the hour, will be the equation of time.

From spherical trigonometry we have

$$1 : \cos A :: \tan AS : \tan AS' = \tan AS \times \cos A;$$

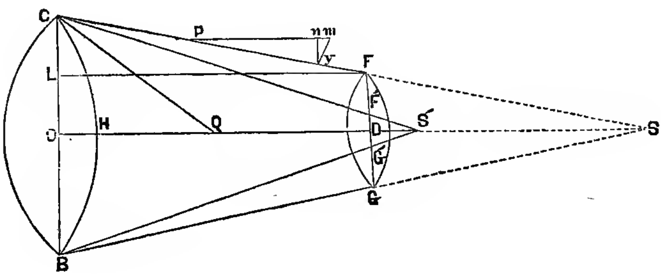
consequently, putting $\tan AS = x$, and
$c = \cos A = \cos 23° \, 28'$ very nearly, we have $\tan AS' = cx$.

Because when $AS - AS'$ is a maximum,

$$\tan (AS - AS') = \frac{\tan AS - \tan AS'}{1 + \tan AS \tan AS'} = \frac{x - cx}{1 + cx^2} = \frac{(1-c)\,x}{1 + cx^2}$$

is also clearly a maximum, it is clear, since c is invariable, that $\frac{1 + cx^2}{x} = \frac{1}{x} + cx$ will be a minimum; consequently (see page 101), we shall have $x = \sqrt{\frac{1}{c}} =$ the tangent of the sun's longitude, and $cx =$ the tangent of the sun's right ascension $= \sqrt{c}$, when the equation of time is a maximum. Because $x \times cx = \sqrt{\frac{1}{c}} \times \sqrt{c} = 1 = 1^2$, it is manifest that we must have $AS + AS' =$ an arc of $90°$; and it is clear from $x = \sqrt{\frac{1}{c}} = 1.04416$, the tan of $46°\ 14'$, that the sun's longitude is $46°\ 14'$, and of course $90° - 46°\ 14' = 43°\ 46'$ must equal his right ascension when the equation of time is a maximum. By subtracting AS' from AS we get $2°\ 28'$ for the maximum value of the equation in degrees, &c., which being converted into time at the rate of $15°$ to the hour, and $15'$ to the minute, &c., gives 9 minutes and 52 seconds, for the maximum value of the equation of time.

19. Supposing the frustum of a right cone whose base and altitude are given, to move forward in a resisting medium in the direction of its length, having its lesser end foremost; to find the diameter of the lesser end, when the resistance of the medium is a minimum.



Let BCFG represent the frustum, $BO = a$ and $G = xD$ the radii of the greater and lesser ends, and $OD = h$ the height. The frustum, moving in the direction OD, it is plain that the resistance of the medium will act in a contrary direction, so that the line mp parallel to OD may stand for the resistance of a particle of the medium, which by drawing mq perpendicular to the side of the frustum, may be resolved into the forces mq and qp, of which the force mq is alone to be retained, since qp acting in the direction of the slant side of the frustum, can not sensibly affect its motion; and, in like manner, by drawing qn at right angles to mp, we may separate the force mq into the two forces mn and nq, of which mn is alone to be retained, since nq is evidently destroyed by an equal and opposite force.

Hence, if pm is represented by unity or (1), it is plain that mn, the only effective part of mp, must be represented by the square of the sine of mpq, the angle of incidence of the resisting particle, with the side of the frustum.

FL being parallel to mp, we may clearly take the angle CFL, whose sine equals $CL \div CF = \frac{a - x}{\sqrt{[h^2 + (a - x)^2]}}$, for the angle of incidence; consequently, $\frac{(a - x)^2}{h^2 + (a - x)^2}$ equals the resistance of each particle that strikes the slant surface of the frustum, while unity equals the resistance of each particle against the smaller end of the frustum.

If x^2 represents the number of particles that strike the lesser end of the frustum, it is plain that $a^2 - x^2$ will represent the number of particles that strike the curve surface of the frustum.

Hence, since the resistance of each particle against the lesser end of the frustum is perpendicular to it and represented by unity, it is plain that x^2 may be taken for the resistance against the lesser end of the frustum, while $\frac{(a-x)^2}{h^2+(a-x)^2} \times (a^2-x^2)$ represents the resistance against the curve surface of the frustum; consequently

$$x^2 + \frac{(a-x)^2(a^2-x^2)}{h^2+(a-x)^2} = \frac{h^2x^2 + a^2(a-x)^2}{h^2+(a-x)^2}$$

$$= a^2 + \frac{(x^2-a^2)h^2}{(a-x)^2+h^2}$$

= the resistance to the whole surface of the frustum = a minimum.

It is hence clear that $\frac{x^2-a^2}{(a-x)^2+h^2}$ must be a minimum, or its reciprocal $\frac{(a-x)^2+h^2}{x^2-a^2}$ must be a maximum.

By putting the differential of this equal to naught, we readily get the equation $\frac{(a-x)^2}{x} = \frac{h^2}{a}$, whose solution gives

$$x = \frac{2a^2+h^2 \mp h\sqrt{4a^2+h^2}}{2a};$$

of which, $$x = \frac{2a^2+h^2 - h\sqrt{4a^2+h^2}}{2a}$$

is clearly the only root that is applicable to the question, since the other root will be greater than the radius of the base of the frustum. From

$$\frac{(a-x)^2}{x} = \frac{h^2}{a} \text{ we have } \frac{(a-x)^2}{h^2} = \frac{x}{a} \text{ or } \frac{CL^2}{FL^2} = \frac{DF}{CO};$$

which, supposing the cone completed, as in the figure, gives $\frac{CO^2}{SO^2} = \frac{SD}{SO}$ or its equivalent $CO^2 = SO \times SD$.

Hence, bisect OD in Q and join QC, and set QC from Q to S′ on QD produced; then, from the right-angled triangle CQO we have

$$CO^2 = CQ^2 - QO^2 = QS'^2 - QD^2$$
$$= (QS' + QD)\,(AS' - QD) = OS' \times S'D,$$

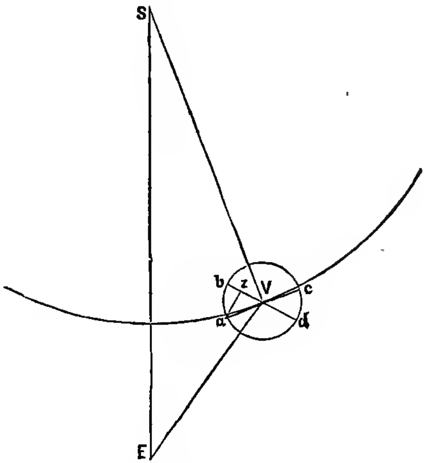
as it ought to do; consequently, the trapezoid CODF′ revolving about OD as an axis, will clearly generate the frustum of minimum resistance.

REMARKS.—1. It will be perceived that the frustum BCFG, has been taken at hazard, and thence from the reasoning the equation $CO^2 = SO \times SD$ found, which has enabled us to find the true frustum, as above.

2. We have taken the example from the scholium to Prop. 34, Sec. 7, Vol. 2, of Newton's "Principia;" and it is easy to perceive that the preceding construction is the same as that of Newton.

.20. To find the position of Venus when brightest, supposing the orbit of the earth and that of Venus to be circles in the same plane, having the sun in their common center.

Let S, E, V, denoting the centers of the sun, earth, and Venus, be connected by right lines forming a triangle; representing the sides SE, SV, and EV, by a, b, and x, and using the circle $abcd$ to represent a section of Venus by the triangle SVE; then, ac and bd being diameters of Venus perpendicular to its distances SV and EV from the sun and earth, it is manifest that ab may be taken to represent the breadth of the illuminated part of Venus, which by drawing az perpendicular to bd gives bz for the versed sine of the angle aVb, when the radius of Venus is represented by unity, which may clearly be taken to vary as the part of Venus that reflects light to the earth at E; consequently, from the principles of optics $\frac{bz}{EV^2}$ will express the quantity of light reflected by Venus to the earth.



From the triangle SEV, we have

$$SE^2=SV^2+EV^2-2SV\times EV \cos SVE=b^2+x^2+2bx \cos aVb$$

since the sum of SVE and aVb is clearly equal to two right angles.

Representing $\cos aVb$ by $\cos \phi$, since $SE^2 = a^2$, we readily get from the preceding equation $\cos \phi = \frac{a^2-b^2-x^2}{2bx}$, which gives

$$\frac{1-\cos \phi}{x^2} = \frac{(b+x)^2-a^2}{2bx^3} =$$

a maximum, since $1-\cos \phi$ represents the versed sine of the angle aVb. Since

$$\frac{(b+x)^2-a^2}{2bx^3} = \frac{(b+x+a).(b+x-a)}{2bx} =$$

a maximum, by taking the hyperbolic logarithm of this, we must have

$$\log(b+x+a)+\log(b+x-a)-3\log x-\log 2b=$$

a maximum; consequently, putting the differential of this equal to naught, we shall have (see p. 54)

$$\frac{1}{b+x+a}+\frac{1}{b+x-a}-\frac{3}{x}=0,$$

or its equivalent $-3b^2-4bx-x^2+3a^2=0$; consequently solving this quadratic, we have $x=-2b+\sqrt{b^2+3a^2}$ and $x=-2b-\sqrt{b^2+3a^2}$, which, giving x negative, must be rejected, and of course we shall have $x=-2b+\sqrt{b^2+3a^2}$. By representing a and b by their proportional distances 1 and 0.72333 nearly, we get, from the preceding equation, $x=0.43046$ for the distance of Venus from the earth. Hence, we easily get SEV $=39°\ 44'$ for the elongation of Venus seen from the earth, and ESV $=22°\ 21'$, the elongation of Venus seen from the sun, which being less than $43°\ 40'$, Venus's greatest elongation, shows that she is brightest between her greatest elongation and her inferior conjunction, being nearly half way between the inferior conjunction and greatest elongation.

Because the preceding reasoning does not give the positions of Venus when she reflects the minimum light, we shall determine these positions after the following method.

Thus, from p. 95, we have

$$F(x-h)=Fx-\frac{d\,(Fx)}{dx}h+d^2\,(Fx)\frac{h^2}{1.2}-, \text{ \&c.},$$

and $$F(x+h)=Fx+\frac{d\,(Fx)}{dx}h+\frac{d^2\,(Fx)}{dx^2}\frac{h^2}{1.2}+, \text{ \&c.};$$

where it will be noticed, that we have shown Fx can not be a maximum or minimum, unless it is determined on the supposition that the term $\frac{d\,(\mathrm{F}x)}{dx}h$ is made to disappear from the equations. Now it is easy to perceive, that we have thus far made the term disappear from the equations by assuming $\frac{d\,(\mathrm{F}x)}{dx}=0$; we now observe that we may, when necessary, make the term disappear from the equations by putting $h=0$. or, since for $\frac{d\,(\mathrm{F}x)}{dx}h$, we may evidently write $\frac{d\,(\mathrm{F}x)}{dx}dx$, we may assume $dx=0$; which, in this question, clearly indicates the inferior and superior conjunctions of the planet; since x is a minimum and maximum at those points.

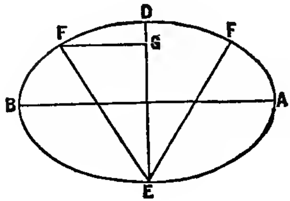
REMARKS.—1. By putting $b=0.3871$, we find $x=1.00058$, and thence get SEV $=22°\ 19'$ for the elongation of Mercury when brightest. Also, the angle ESV $=78°\ 56'$, while it is only $67°\ 13'.5$ at the time of the planet's greatest elongation; consequently Mercury is brightest between its greatest elongation and the superior conjunction.

2. Because the motion of Venus about the sun relatively to that of the earth is about 37′; by dividing $22°\ 21'=1341'$ by 37′, we get 36 days for the time when Venus is brightest before and after her inferior conjunction.

3. If we apply the formula $x=-2b+\sqrt{b^2+3a^2}$, to find the position of a superior planet when brightest, it will be found to be impossible; for, since $3a^2$ will be less than $3b^2$, it follows that $\sqrt{b^2+3a^2}$ will be less than $2b$, and, of course, $x=-2b+\sqrt{b^2+3a^2}$ will be negative, which is impossible, since x, the distance of the planet from the earth, is always positive. Hence, it is manifest that $\frac{d\,(\mathrm{F}x)}{dx}dx=0$, when applied to the superior planets, can only be satisfied

by putting $dx = 0$; which clearly indicates that they reflect the most light in their oppositions, and the least in their conjunctions with the sun.

21. From the extremity of the minor axis of an ellipse to draw the maximum line to the opposite part of its perimeter.



Let AEB be an ellipse, having AB and DE for its major and minor axes; and let EF be the maximum line required, having EG and FG for the rectangular co-ordinates of its extremity F. Then, a and b representing the major and minor axes, we have $b^2 : a^2 :: bx - x^2 : y^2 = \frac{a^2}{b^2}(bx - x^2)$, from a well-known property of the curve. Hence, adding x^2 to y^2, we have $EF^2 = EG^2 + FG^2 = x^2 + y^2 = x^2 + \frac{a^2}{b^2}(bx - x^2)$; consequently, since EF^2 is a maximum, $x^2 + \frac{a^2}{b^2}(bx - x^2)$ must be a maximum; and putting its differential equal to naught, we have $2x + \frac{a^2}{b^2}(b - 2x) = 0$, which gives $x = \frac{1}{2}\frac{a^2 b}{a^2 - b^2}$, which is clearly a maximum, since the differential of $2x + \frac{a^2}{b^2}(b - 2x)$ is $\left(2 - \frac{2a^2}{b^2}\right)dx$, which is negative, when dx is positive, as it ought to be.

REMARKS.—1. It is evident from the nature of the ellipse, that the relation of its axes must be such that x shall not be greater than the minor axis b, in order that the preceding value of x may be applicable to it; since EG must clearly not be greater than ED $= b$.

Hence, to find the greatest value that b can have when the preceding solution is possible, we put b for x in $x = \frac{1}{2}\frac{a^2 b}{a^2 - b^2}$, and thence get $2b^2 = a^2$ or $b = \frac{a}{\sqrt{2}}$; consequently, when b has this or a less value, the maximum line will be found from $x = \frac{1}{2}\frac{a^2 b}{a^2 - b^2}$.

The question and the substance of what has been said are substantially the same as given by T. Simpson, at pages 35 and 36 of his "Fluxions," for the purpose of showing whether the *solution found in any case* falls within the limits required by the nature of the question.

2. Resuming, $EF^2 = x^2 + \frac{a^2}{b^2}(bx - x^2)$, and putting its differential equal to naught, we have $\left(2x + \frac{a^3}{b^2}(b - 2x)\right) dx = 0$; which, by putting $dx = 0$, clearly gives the minor axis ED for the maximum line when the minor axis is not less than $\frac{a}{\sqrt{2}}$. When the minor axis is not greater than $\frac{a}{\sqrt{2}}$, by putting the factor $2x + \frac{a^2}{b^2}(b - 2x)$ of the preceding differential equal to naught, we get $x = \frac{1}{2}\frac{a^2 b}{a^2 - b^2}$, to determine the maximum value of the line to be drawn; and by putting $dx = 0$, we get the minor axis ED for the minimum value, as is manifest from the consideration that when b is less

than $\frac{a}{2}$ there will be two maxima values, represented by EF and EF′, and of course ED must be a minimum.

It is hence clear that the (common) rule for finding the maxima and minima of a function of a single variable, given at p. 96, is not always sufficiently general.

22. Given $\quad x + y + z = a \quad$ and $\quad x^m y^n z^p$

or $\quad m \log x + n \log y + p \log (a - x - y)$

a maximum, to find x, y, and z.

It is manifest that (since x and y are independent variables) we may put the differential of the preceding equation, with reference to y, equal to nothing, and thence get y in terms of x; by which means we shall reduce the question to that of making a function of a single variable equal to naught.

Thus, we shall have

$$\frac{ndy}{y} - \frac{pdy}{a - x - y} = 0 \quad \text{or} \quad \frac{n}{y} - \frac{p}{a - x - y} = 0,$$

which gives $y = \frac{n(a - x)}{n + p}$; consequently, putting this value for y in $m \log x + n \log y + p \log (a - x - y)$, it becomes

$$m \log x + n \log (a - x) + \log n^n$$

$$- (p + n) \log (n + p) + p \log (a - x) + \log p^p,$$

which must be a maximum; consequently, putting its differential equal to naught, we have

$$\left(\frac{m}{x} - \frac{n + p}{a - x}\right) dx = 0,$$

whose differential gives

$$- \left(\frac{m}{x^2} + \frac{n + p}{(a - x)^2}\right) \times dx^2 = 0.$$

From the first of these equations we get

$$x = \frac{am}{m+n+p},$$

and the second equation shows $x^m y^n z^p$ to be a maximum, as required.

Remarks.—1. This example has been before solved at page 100, and it is easy to perceive that we obtain the same results as there found, by substituting the value of x in that of y, and then substituting the values of x and y for them, in $z = a - x - y$.

2. We have here given the preceding solution of it, for the purpose of showing the facility with which a function of any number of independent variables may be made a maximum or minimum, by reducing it to the maximum or minimum of a function of a single variable; since it is easy to perceive that we may proceed in like manner, whatever may be the number of independent variables.

To make what is here said more clear, we will apply the process to the following example from page 33 of Simpson's "Fluxions."

23. To find such values of x, y, and z, as shall make $(b^3 - x^3)\ (x^2z - z^3)\ (xy - y^2)$ a maximum.

From making y alone variable, we have $xdy - 2ydy = 0$; which gives $y = \frac{x}{2}$, and thence $xy - y^2 = \frac{x^2}{4}$.

In like manner, making z alone variable in the proposed equation, we have $x^2dz - 3x^2dz = 0$; which gives $z = \frac{x}{\sqrt{3}}$ and thence $x^2z - z^3 = \frac{2x^3}{3\sqrt{3}}$. By substituting the preceding values in the proposed expression, it becomes

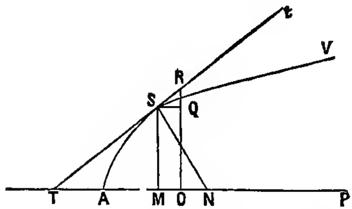
$$(b^3 - x^3) \times \frac{2x^3}{3\sqrt{3}} \times \frac{x^2}{4} = \text{maximum};$$

consequently, we must make $b^3x^5 - x^8$ a maximum.

Hence we have $5b^3x^4dx - 8x^7dx = 0$, and $(20b^3x^3 - 56x^6)dx^2$; the first of these gives $x = \frac{b}{2}\sqrt[3]{5}$, which put for x in the second expression makes it negative, and of course shows the proposed expression to be a maximum as required.

SECTION V.

TANGENTS AND SUBTANGENTS, NORMALS AND SUBNORMALS, ETC.



(1.) Suppose **AM** $= x$ **and MS** $= y$ **to** stand for the abscissa and ordinate of the point S of the plane curve ASV, having A for their origin; then, conceiving the curve to be described by the extremity S of the ordinate, while the other extremity M moves uniformly from the origin A of the coordinates along the line of the abscissæ toward P, so as to keep the ordinate constantly parallel to itself during the motion, we may clearly suppose the ordinate to increase or decrease in such a way as to describe the curve.

(2.) If now we suppose the right line TSt to pass through the extremity S of the ordinate of the curve ASV in any one of its positions, in such a way that the first differential coefficient of the equation of the curve equals the first differential coefficient of the right line; then, *the right line is said to touch the curve, or the line and curve are said to touch each other, and the point* S *is called their points of contact.*

Thus, if $y = \phi(x)$ and $y = Ax + B$ represent the equations of the curve and right line, by taking their differential coefficients, we shall have $\frac{dy}{dx} = \frac{d\phi(x)}{dx}$ and $\frac{dy}{dx} = A$; consequently, according to the definition, we shall have $A = \frac{d\phi(x)}{dy}$; which, when x is known, will give the value of A, and thence the tangent can easily be drawn.

Since $A = \frac{dy}{dx}$ is derived from the equation of the curve, if x and y represent the co-ordinates of the point of contact of any tangent with the curve, and X and Y the co-ordinates of any other point of the tangent, it is manifest that

$$Y - y = \frac{dy}{dx}(X - x)$$

may be taken as the general equation of the tangent.

Putting $Y = 0$, in this equation, we easily get

$$\frac{y}{\frac{dy}{dx}} = x - X = MT,$$

called *the subtangent*, which is known, since y and $\frac{dy}{dx}$ are known from the equation of the curve; noticing, since x and y are supposed to be taken in the direction of the positive co-ordinates, that $X = -AT$ taken in the direction of x negative, must be subtracted from $x = AM$ taken in the direction of x positive, which gives $x - X = AM + AT = MT$, as above.

It may be proper to illustrate what is here done, in another way: thus, let the ordinate of the right line RO be drawn very near SM, SQ be drawn parallel to MO or the axis of x; then, representing MO or SQ by dx, it is clear that QR will represent dy both in the right line and curve; and that the triangles SRG and STM, being equiangular, give (from well-known principles of geometry) the proportion

$$RQ : SQ :: SM : MT,$$

or its equivalent $\frac{dy}{dx} = SM \div MT$; which gives $\frac{y}{\frac{dy}{dx}} = TM$, or $\frac{dxy}{dy} = MT$, which agree with the expression for the subtangent before found. Where, it may be added, that if the co-ordinates are at right angles to each other,

$$\frac{dy}{dx} = \text{tan ang RSQ} = \text{tan ang STM}$$

to radius unity; consequently, *we find the subtangent by dividing the ordinate by the tangent of the angle of the inclination of the tangent to the line of the abscissæ*, when the axes of the proposed curve are rectangular, or, which comes to the same, *we multiply the ordinates by the tangent of the angle it makes with the tangent, for the subtangent.*

(3.) If SN is drawn through the point of contact S, perpendicular to the tangent, meeting the axis of x in N, it will be what is called *the normal;* particularly when AP is the axis of the curve, or cuts the tangent at A perpendicularly, and the ordinates of the curve are perpendicular to the axis; also, MN is called *the subnormal.*

Since the triangles SRQ and STM are equiangular, it is manifest that whatever may be the angle AMS, provided it is known, we can always find all the parts of the triangle STM from the equation of the curve and knowing the ordinate SM, since $SM = y$ gives $TM = y \div \frac{dy}{dx} =$ the subtangent. Hence, we easily get SN, the normal, from the triangle STN, and TN from the same triangle; consequently, TN — TM = MN = the subnormal is found.

We shall, in what is to follow (according to custom), suppose AP to be the axis of the curve, whose ordinate y cuts it perpendicularly; then, the triangle SRQ will evidently be similar to the triangle SMN, and will give

$$SQ : RQ :: SM : MN, \text{ or } dx : dy :: y : MN = \frac{ydy}{dx}$$

and thence the normal

$$SN = \sqrt{\left(y^2 + y^2\frac{dy^2}{dx^2}\right)} = y\sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}$$

is found: noticing, that in much the same way from the triangle STM, we get the tangent $ST = y\sqrt{\left\{1 + \sqrt{\left(\frac{dx}{dy}\right)^2}\right\}}$.

There is another way of finding the subnormal from the equation of the normal, which it may not be improper to notice in this place.

Thus, assuming $Y - y = A(X - x)$ to represent the equation of the normal, we shall have A equal to the tangent of the ang. SNP to radius unity; observing, that the angles which right lines make with the axis of x, are supposed to be included between them and x positive, estimated (according to usage) from right to left.

Because the angle SNP, from well-known principles of geometry, equals the sum of the inward and opposite angles, $S = 90°$ and T of the triangle NST, we shall have

$$\tan SNP = \tan(90° + T) = \frac{\sin(90° + T)}{\cos(90° + T)} = \frac{\cos T}{-\sin T}$$

$$= -\frac{1}{\tan T} = -\frac{1}{\frac{dy}{dx}} = -\frac{dx}{dy}; \text{ see pp. 62 and 63.}$$

Hence, from the substitution of the value of A in the equation of the normal, it will become

$$Y - y = -\frac{1}{\frac{dy}{dx}}(X - x) = -\frac{dx}{dy}(X - x);$$

which is the well-known equation of the normal.

If in this we put $Y = 0$, we shall have $-y = -\frac{dx}{dy}(X - x)$; or, since $X - x = AN - AM = MN$, the subnormal $= \frac{ydy}{dx}$, the same as before found.

(4.) Resuming the equations that have been found, when the ordinates are perpendicular to the line of the abscissæ, we shall have $$Y - y = \frac{dy}{dx}(X - x) \ldots\ldots\ldots\ldots (1)$$ for the equation of the tangent that passes through the point (x, y) of any plane curve; and

$$Y - y = -\frac{1}{\frac{dy}{dx}}(X - x) = -\frac{dx}{dy}(X - x) \ldots\ldots (2)$$

is the equation of the normal to any plane curve, at the point (x, y): noticing, that these formulas clearly show that to find the tangent or normal at any point (x, y) of any plane curve, it will be necessary to find $\frac{dy}{dx}$ from the equation of the curve, and to put its value in the preceding equations.

Thus, to find the tangent and normal to the circle whose equation is $y^2 + x^2 = r^2$, r being the radius.

By taking the differential, we have $2ydy + 2xdx = 0$; which gives $\frac{dy}{dx} = -\frac{x}{y}$; and substituting this value of $\frac{dy}{dx}$ in (1) and (2), we have $Y - y = -\frac{x}{y}(X - x)$, or, by a simple reduction, $Yy + Xx = y^2 + x^2 = r^2$; and $Y - y = \frac{y}{x}(X - x)$, or $Yx = yX$, which shows that the normal passes through the center of the circle at right angles to the tangent.

Taking $a^2y^2 + b^2x^2 = a^2b^2$, the equation of the ellipse, and proceeding as before, we get $a^2Yy + b^2Xx = a^2b^2$, and $a^2Xy - b^2Yx = (a^2 - b^2)yx$, for the equations of the tangent and normal, which are well-known forms.

Supposing a and b to be the half major and minor axes of the ellipse, by putting Y and X successively equal to naught in the preceding equations, we have

$$X = \frac{a^2}{x} \text{ and } Y = \frac{b^2}{y}, \quad \text{and} \quad X = \frac{a^2 - b^2}{a^2}x, \quad Y = \frac{a^2 - b^2}{-b^2}y;$$

which are well-known forms for drawing tangents and normals to the ellipse.

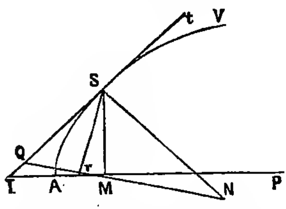
REMARKS.—1. Because $a^2y^2 - b^2x^2 = -a^2b^2$, the equation of the hyperbola, is deduced from that of the ellipse by changing the signs of its terms that involve b^2; it is clear that, by changing the signs of the terms that involve b^2 in the above results, they will give $X = \frac{a^2}{x}$ and $Y = \frac{-b^2}{y}$, and $X = \frac{a^2 + b^2}{a^2}x$, $Y = \frac{a^2 + b^2}{b^2}y$, for the corresponding quantities in the hyperbola.

2. From $X = \frac{a^2}{x}$ and $Y = -\frac{b^2}{y}$, by supposing x and y to be infinitely great, it is clear that X and Y will become unlimitedly small, or that tangents to the hyperbola at points infinitely remote from the center will pass through it very nearly, and have $a^2y^2 - b^2x^2 = 0$, or its equivalents $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$, the equations of right lines passing through

the center of the hyperbola, for their limits; in such a sense, that they will ultimately differ insensibly from coinciding with these equations: noticing, that these limits are called *the asymptotes* of the hyperbola; and from $a^2y^2 - b^2x^2 = 0$ and $a^2y^2 - b^2x^2 = -a^2b^2$, it is also clear that the asymptotes may be regarded as exterior limits of the hyperbola.

3. It is easy to perceive that the preceding conclusions with reference to tangents and asymptotes are independent of the angle formed by the axes of co-ordinates; provided a and b represent a pair of semiconjugate diameters having the same directions as the axes of co-ordinates.

(5.) We will now show how to draw tangents and normals to plane curves, referred to polar co-ordinates.



Thus, let $rS = r$ be the radius vector drawn from the pole r to any point S of the curve, ArP the angular axis making the angle PrS $= \omega$ with the radius vector; then through the pole draw NrQ at right angles to the radius vector, meeting the tangent to the curve at S in Q and the normal SN to the curve at the same point in N; then we shall take rQ and rN for the subtangent and subnormal of the tangent SQ and normal SN to the curve at S. Drawing SM $= y$ perpendicular to the angular axis AP, we may evidently take $r\text{M} = x$ and SM $= y$ for the rectangular coördinates of the point S of the proposed curve, having r

for their origin; and from the right-angled triangle rSM, we get, from the well-known principles of trigonometry, $r \sin \omega = y$ and $r \cos \omega = x$, which give $r^2 = x^2 + y^2$.

Because the angle SrP $= \omega$ is the exterior angle of the triangle STr, we shall have the angle $S = \omega - T$, which gives $\tan S = \tan(\omega - T) = \frac{\tan \omega - \tan T}{1 + \tan \omega \tan T}$, or since $\tan \omega = \frac{y}{x}$ and $\tan T = \frac{dy}{dx}$ (page 127), we have

$$\tan S = \left(\frac{y}{x} - \frac{dy}{dx}\right) \div \left(1 + \frac{ydy}{xdx}\right) = \frac{ydx - xdy}{xdx + ydy}$$

$$= \frac{y^2 d\frac{x}{y}}{\frac{1}{2} dr^2} = \frac{r^2 \sin^2 \omega \, d \cot \omega}{rdr} = -\frac{rd\omega}{dr}.$$

Hence, since $\tan S = -\frac{rd\omega}{dr}$ and $\cot S = -\tan r\text{SN}$, we get from the triangles rSQ and rSN, by trigonometry,

$$r\text{Q} = \text{S}r \times \tan S \quad \text{or} \quad r\text{Q} = -\frac{r^2 d\omega}{dr},$$

and $$r\text{N} = -\text{S}r \times \cot S = -r \div -r\frac{d\omega}{dr} = -\frac{dr}{d\omega},$$

for the expressions for the subtangent and subnormal, as required.

It may be added, that if we regard x and y as functions of ω, we may for dx and dy in $\tan S = \frac{ydx - xdy}{xdy + ydy}$, evidently write $\frac{dx}{d\omega}$ and $\frac{dy}{d\omega}$; consequently, from $x = r \cos \omega$ and $y = r \sin \omega$, we readily get

$$\frac{dx}{d\omega} = \frac{dr}{d\omega} \cos \omega - r \sin \omega \quad \text{and} \quad \frac{dy}{d\omega} = \frac{dr}{d\omega} \sin \omega + r \cos \omega.$$

Hence, from the substitution o^ these values of

$$\frac{dx}{d\omega} \text{ and } \frac{dy}{d\omega} \text{ in } \tan S = \frac{ydx - xdy}{xdx + ydy},$$

we get, after obvious reductions,

$$r \tan S = -\frac{r^2}{\frac{dr}{d\omega}} = -\frac{r^2 d\omega}{dr}, \text{ and thence } rN = -\frac{dr}{d\omega}, \text{ the same}$$

as before; see p. 117 of Young's "Differential Calculus."

Since the angles SrP and $SrT = 180°$, if we represent SrT by θ, we shall get $\omega = 180° - \theta$, which gives $d\omega = -d\theta$. Hence, the preceding equations become

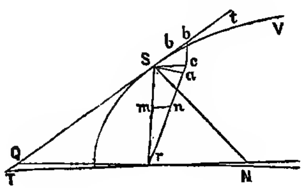
$$r \tan S = r \times \frac{rd\theta}{dr} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3)$$

and

$$r \cot S = r \tan rSN = \frac{dr}{d\theta}. \quad . \; . \; . \; . \; . \; . \quad (4):$$

noticing, that θ, the angle TrS, is the length of an arc of a circle whose radius is unity and center r, which is measured from rT toward rS, and may contain one or more circumferences, or any part of the circumference, according to the nature of the case.

Remark.—The substance of what has been done may be expressed in the following simple manner.



Thus, from r draw the right line rb to make the small angle $d\theta$ with rS, and from r as a center, with radii $rm = 1$ and $rS = r$, describe the small arcs mn and Sa, cutting rS

and rb in the points mn and Sa; then, $mn = d\theta$, and from the similarity of the circular sectors, we have

$$1 : d\theta :: r : \text{S}a = rd\theta.$$

Then, through S draw Sc perpendicular to rS and equal to $rd\theta$, and cb' perpendicular to Sc, meeting the tangent Tt in b'; then, the triangle Scb' is evidently equiangular to the triangles rSQ and rSN. From the equiangular triangles Scb' and SrQ, we have the proportion $b'c$: Sc :: Sr : rQ, or $r\theta = b'c : rd\theta :: r : r\text{Q} = \frac{rd\theta}{b'c} \times r$; consequently, by comparing this value of rQ with that of (3) at p. 133, we must have $b'c = dr$; and of course if $d\theta$ represents the differential of θ, taken for the independent variable, $b'c$ must represent dr, the differential of r; r being a function of θ, see (4.) at p. 2. We also have, from the triangles Scb' and SrN, the proportion Sc : $b'c$ :: Sr : rN, or $rd\theta : dr :: r : r\text{N} = \frac{dr}{d\theta}$, which is the same as (4) at p. 133.

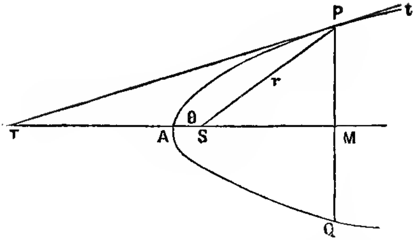
It is hence evident, that if the angle $d\theta$ is infinitely small, the triangle Scb' and the curve line triangle Sab will come infinitely near to coincidence; consequently, according to the method of *limiting ratios*, we shall ultimately have

$$ba : \text{S}a :: \text{S}r : r\text{Q} \quad \text{or} \quad dr : rd\theta : r : r\text{Q} = \frac{r^2 d\theta}{dr}$$

for the subtangent, and $rd\theta : dr :: r : r\text{N} = \frac{dr}{d\theta}$ for the subnormal; results in conformity to the method of limiting ratios; see p. 46.

To illustrate what has been done, we will show how to draw tangents and normals to the common parabola, whose equation is $4ax = y^2$, when the pole is taken at the focus.

Thus, let PAQ represent the parabola, having AM for its



axis and A for its vertex; then S, being the focus, we have $4a = 4AS$, and representing the perpendiculars AM and PM by x and y, the equation $4AS \times AM = PM^2$ of the parabola, becomes $4ax = y^2$; which expresses the equation referred to rectangular co-ordinates. Supposing the right line Tt touches the parabola at P and intersects the axis in T, we have $SP = r$ and the angle PST included by PS and $TS = \theta$. By trigonometry, we have

$$PS \cos \text{ang } PSM = -r \cos \theta = SM = x - a,$$

or $x = a - r\cos\theta$, and $PS \times \sin PSM = r \sin \theta = y$;

consequently, from the substitution of these values in the equation $4ax = y^2$, we have $4a(a - r\cos\theta) = r^2 \sin^2\theta$, which is easily reduced to the form

$$4a^2 - 4ar\cos\theta + r^2\cos^2\theta = (2a - r\cos\theta)^2$$
$$= r^2\sin^2\theta + r^2\cos^2\theta = r^2 \text{ (since } \sin^2\theta + \cos^2\theta = 1);$$

consequently, we readily get $r = \dfrac{2a}{1 + \cos\theta} = \dfrac{a}{\cos^2 \dfrac{\theta}{2}}$ for the

polar equation. By taking the differentials of this equation,

we have $dr = \dfrac{a \sin \dfrac{\theta}{2} d\theta}{\cos^3 \dfrac{\theta}{2}}$; and thence $\dfrac{rd\theta}{dr} = \cot \dfrac{\theta}{2}$, which, substituted for (3) in p. 133, gives

$$r \cot \frac{\theta}{2} = \text{PS} \times \tan \text{ang SPT}$$

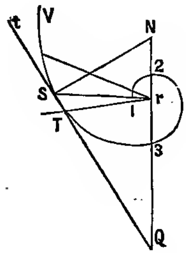
= the perpendicular from S to SP, limited by the tangent PT = the sought subtangent; and from (4), at p. 133, we have $\frac{dr}{d\theta} = \frac{a \sin \frac{\theta}{2}}{\cos^3 \frac{\theta}{2}}$ = the subnormal $= r \tan \frac{\theta}{2}$ = the perpendicular to SP through S, produced to meet the perpendicular to T*t* through P, which gives the limit of the required normal.

REMARKS.—1. By taking the differentials of the members of the equation $4ax = y^2$, we have $4adx = 2ydy$, which gives $\frac{dx}{dy} = \frac{2y}{4a}$, and thence $y \times \frac{dx}{dy} = \frac{2y^2}{4a} = 2x$ = the subtangent, agreeably to what is shown at page 127; consequently, by taking MT $= 2\text{AM} = 2x$, and joining P and T by a right line, it will touch the parabola at P.

2. From $4adx = 2ydy$, we get $2a = \frac{ydy}{dx}$ = subnormal (see p. 128), is constant, and equal to $\frac{4a}{2}$ = half the parameter of the axis of the parabola; since $4a$ is called *the parameter* or *latus rectum* of the axis of the parabola.

For another example, we will show how to draw the tangent and normal to the logarithmic spiral, whose equation is $r = a^\theta$; by using polar co-ordinates.

Let 1, 2, 3, V, represent the spiral, having r for its pole, and r, 1, T for its angular axis, such that the positive values of θ are the arcs of a circle (rad. = 1), which increase arithmetically in the order 1, 2, 3, V, while $r = a^\theta$ increases geo-



metrically; then, because $\theta = 0$ at 1, it is clear that since $r = a^{\theta} = a^{0} = 1$, that we must have $r = 1$ represented by $r1$, and in $r = a'$, $\theta = 1$ must be represented by the arc of the circle whose length is 1, which we may suppose to equal the length of its radius.

By taking the hyperbolic logarithms of the members of $r = a^{\theta}$, we have $\log r = \theta \log a$; whose differentials give $\frac{dr}{r} = \log a \, d\theta$, or $\frac{dr}{rd\theta} = \tan \text{ ang } r\text{ST} = \log a$; consequently, the angles at which the radius vector cuts the spiral having the constant, $\log a$, for its tangent, must be constant or invariable; noticing, if $\log a = 1$, that the radius vector cuts the spiral at an angle of 45° or half a right angle.

By (3) and (4), given at page 133, if we divide r by $\frac{dr}{rd\theta} = \log a$, and multiply r by $\frac{dr}{rd\theta} = \log a$, we shall have $\frac{r}{\log a} = ra$, and $r \log a = r\text{N}$, for the subtangent and subnormal, as required; consequently, the tangent and normal can readily be drawn.

(6.) When a right line touches a curve at an infinitely remote point from the origin of the co-ordinates, and at the same time passes at a finite distance from the origin of the co-ordinates, it is said to be a *rectilinear asymptote to the curve.*

Thus, by assuming the equation of the tangent from (1), given at page 129, we have $Y - y = \frac{dy}{dx}(X - x)$, in which x and y belong to the point of contact of the tangent with the curve, while X and Y are the co-ordinates of any other point of the tangent; then, by putting Y and X successively equal to naught, we have

$$-y = \frac{dy}{dx}(X - x), \quad \text{or} \quad X = x - \frac{ydx}{dy} \;.\;.\;.\;.\; (5),$$

and

$$Y = y - x\frac{dy}{dx} \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\; (6);$$

from which it clearly results, if X and Y, or either of them, is finite when x or y is infinite, that the curve must have a rectilineal asymptote, while if X and Y are both infinite or impossible, the curve has no rectilineal asymptote.

Thus, from $y = \frac{b}{a}\sqrt{x^2 - a^2}$, the equation of the hyperbola,

we have $\quad ay = \frac{b}{a}xdx \div \sqrt{(x^2 - a^2)} \quad$ or $\quad \frac{dx}{dy} = \frac{\sqrt{x^2 - a^2}}{\frac{b}{a}x}$,

which with the value of y reduce (5) to $X = x - \frac{x^2 - a^2}{x}$; which, by making x infinite, and rejecting a^2 on account of its minuteness, with reference to x^2, becomes

$$X = x - \frac{x^2}{x} = x - x = 0;$$

consequently, the hyperbola has an asymptote passing through the center.

Again, reducing the equation to $x = \frac{a}{b}\sqrt{(b^2+y^2)}$, we have $\frac{dy}{dx} = \frac{b}{a}\frac{\sqrt{(b^2+y^2)}}{y}$, and thence (2) is readily reduced to $Y = y - \frac{b^2+y^2}{y}$; consequently, making y infinite and rejecting b^2 on account of its comparative smallness, we have $Y = y - \frac{y^2}{y} = y - y = 0$, and of course, as before, the curve has an asymptote passing through the center.

Resuming the equation $y = \frac{b}{a}\sqrt{x^2 - a^2}$, and supposing x unlimitedly great, by rejecting a^2 on account of its comparative smallness, we have $y = \frac{b}{a}x$; or, since the radical ought to have the ambiguous sign $\pm$, we get $y = \pm\frac{b}{a}x$, or its equivalents, $y = \frac{bx}{a}$ and $y = -\frac{bx}{a}$; which clearly are the equations of two right lines that are asymptotes to the hyperbola, passing through the center of the curve, in accordance with what has been before shown; noticing, that equivalent conclusions result immediately from

$$\frac{dy}{dx} = \pm\frac{b}{a}\frac{x}{\sqrt{(x^2-a^2)}} \quad \text{and} \quad \frac{dy}{dx} = \frac{b}{a}\frac{\sqrt{b^2+y^2}}{y},$$

making x infinite in the first and y in the second, and rejecting a^2 in comparison to x^2, and b^2 in comparison to y^2; and we thus get $dy = \pm\frac{b}{a}dx$, or its equivalents $dy = \frac{b}{a}dx$ and $dy = -\frac{b}{a}dx$, which are clearly the differentials of the equations given above.

REMARKS.—1. If we convert

$$y = \frac{b}{a}\sqrt{(x^2 - a^2)} = \frac{bx}{a}\sqrt{\left(1 - \frac{a^2}{x^2}\right)} = \frac{bx}{a}\sqrt{(1 - a^2x^{-2})}$$

into a series arranged according to the descending powers of x, we shall have

$$y = \frac{bx}{a}\left(1 - \frac{a^2x^{-2}}{2}\right) - \frac{a^4x^{-4}}{8} - \frac{a^6x^{-6}}{16} -, \text{ \&c.};$$

which, when x is very great, clearly gives

$$y = \pm\frac{bx}{x}, \quad y = \pm\frac{bx}{a}\left(1 - \frac{a^2x^{-2}}{2}\right),$$

$$y = \pm\frac{bx}{a}\left(1 - \frac{a^2x^{-2}}{2} - \frac{a^4x^{-4}}{x}\right), \text{ \&c.},$$

for a succession of lines that are clearly asymptotes to each other, and to the hyperbola; noticing, that these are sometimes called *hyperbolic asymptotes*, because the first of them are right lines. It is evident from

$$x = \frac{a}{b}\sqrt{(y^2 + b^2)} = \frac{ay}{b}\sqrt{(1 + b^2y^{-2})},$$

that we may express the asymptotes in terms of the descending powers of y.

2. From $a^2y^2 = x^4 - b^4 = x^4(1 - b^4x^{-4})$, we in like manner get

$$y = \frac{x^2}{a}\left(1 - \frac{b^4x^{-4}}{2} - \frac{b^8x^{-8}}{8} - \frac{b^{12}x^{-12}}{16} -, \text{ \&c.}\right)$$

which gives $y = \pm\frac{x^2}{a}, \quad y = \pm\frac{x^2}{a}\left(1 - \frac{b^4x^{-4}}{2}\right),$

$$y = \pm\frac{x^2}{a}\left(1 - \frac{b^4x^{-4}}{2} - \frac{b^8x^{-8}}{8}\right), \text{ \&c.},$$

for asymptotes to the curve whose equation is $a^2y^2 = x^4 - b^4$, and to each other; and because none of these are rectilineal, they are from their forms said to be *parabolic asymptotes.*

3. It is hence easy to perceive how we may proceed to find the asymptotes of curves that admit of them. Thus, to find asymptotes of the curve whose equation is

$$y^3x^2 - px^3 - bx + c = 0.$$

Dividing its terms by x^2, the equation is reduced to

$$y^3 - px - bx^{-1} + cx^{-2} = 0;$$

consequently, $y^3 - px = 0$, and $y^3 - px - bx^{-1} - 0$, are the successive parabolic asymptotes of each other and the proposed curve.

If we take the curve whose equation is "$my^3 - x^2y = mx^3$," and develop y into a series of the descending powers of x, we shall in like manner get $y = -m$, $y = -m - m^4x^{-3}$, $y = -m - m^4x^{-3} - 3mx^{-6}$, &c., for the hyperbolic asymptotes of the proposed curve; noticing, that the first of these is a right line parallel to the axis of x on the side of y negative, drawn at the distance m, from x.

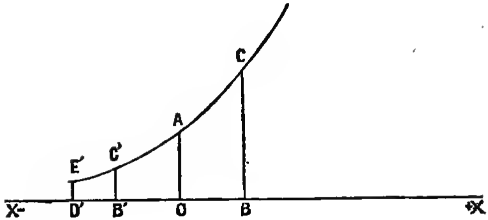
4. When a curve is referred to polar co-ordinates, it is clear that there will always be an asymptote when r, the radius vector, is infinite, and the corresponding value of θ is finite; but if r and θ are both infinite, there is no asymptote.

(7.) To illustrate what has been done more fully, take the following

EXAMPLES.

1. To draw a tangent and normal to any point of the logarithmic curve, and to determine its asymptote.

Let 0ACB be the logarithmic curve, having 0 for the origin of its rectangular co-ordinates, and 0A, BC, &c., for its ordinates on the side of x positive, and B′C′, D′E′, &c., on the side of x negative; then, $y = a^x$, representing the equation of the curve, by taking the hyperbolic logarithms of its members, we shall have $\log y = x \log a$, so that x,



being supposed to commence at 0, we shall have $0A = 1$ for the unit of length, and $\log BC = 0B \times$ the hyperbolic logarithm of a, and, changing the sign of x, we have $\log B'C' = -0B' \times \log a$, and so on. By taking the differentials of the members of the equation $\log y = x \log a$, we have $\frac{dy}{y} = dx \log a$, which gives $\frac{ydx}{dy} = \frac{1}{\log a} = m =$ the subtangent $=$ const, and $\frac{ydy}{dx} = y^2 \log a$, the subnormal [see (1) and (2)], which is clearly correct, since y is a mean proportional between the subtangent and subnormal; hence, joining the point of contact of the tangent with the extremities of the subtangent and subnormal, the tangent and normal required become known.

To find the asymptote: by changing the sign of x, the equation $y = a^x$ becomes $y = a^x = \frac{1}{a^x}$; consequently, since a is supposed to be positive and sensibly greater than unity, it clearly follows, from $y = \frac{1}{a^x}$, that if x is unlimitedly great, y is unlimitedly small, and thence the axis of x is plainly an asymptote to the curve.

Remark.—It is evident, from what is here done, and from what has been done at p. 136, that the logarithmic curve and the logarithmic spiral, have resulted from different ways of expressing the relation of a system of numbers and their logarithms by linear description.

2. To draw a tangent to the curve whose equation is $xy = A^2$, and find its asymptotes.

By taking the differential of the members of the equation, since $A^2 = \text{const}$, we have $ydx + xdy = 0$, which gives $\frac{ydx}{dy} + x = 0$ or $\frac{ydx}{dy} = -x =$ the subtangent; which, being negative, shows that it lies in a contrary direction from what has heretofore been supposed, or that it falls in the direction of the positive values of x. If the extremity of the subtangent is joined with the point of contact of the tangent and curve, the tangent will of course be drawn as required. Because the proposed equation is equivalent to either of the forms $y = \frac{A^2}{x}$ or $x = \frac{A^2}{y}$; it is clear, from the first form, that making x infinitely great, reduces y to an infinitesimal; and, from the second form, it results that indefinitely great values of y give infinitesimal values of x; consequently, the axes of x and y are asymptotes of the curve.

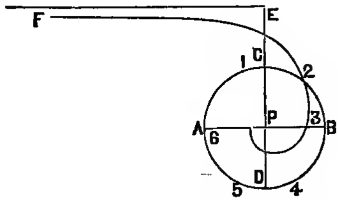
REMARK.—It is easy to perceive, that the equation $xy = A^2$ is that of the hyperbola, when referred to its asymptotes as axes of co-ordinates.

3. To find the subtangent in the hyperbolic spiral, whose equation is $r\theta = a$.

By taking the differentials, as in the preceding example, we have $rd\theta + \theta dr = 0$; which gives

$$\frac{rd\theta}{dr} = -\theta, \quad \text{and} \quad \frac{r^2 d\theta}{dr} = -r\theta = -a = \text{the subtangent}$$

[see (3) at p. 133]; consequently, the subtangent is negative and equal to a. It is hence easy to perceive how the curve may be constructed, and its asymptote drawn, since it manifestly has an asymptote.



Thus, describe the circle ACBD with the unit of distance as radius, and draw the perpendicular diameters AB and CD, taking the first of them for the angular axis and the center P of the circle for the pole of the spiral; then, producing PC to E, such that PE $= a$, and drawing EF perpendicular to it, EF will clearly coincide in direction with the asymptote, on the supposition that the values of θ in the equation $r\theta = a$ are estimated from A, in the order of the letters ACBD, to include any number of revolutions that may be desired.

To describe the spiral by points; we put its equation in the form $r = \frac{a}{\theta}$, and thence, since the semicircumference of the circle ACB $= \pi = 3.1416$ very nearly, from knowing a, we readily find the corresponding values of r.

Thus, by taking the arc A1 equal to the radius of the circle $=$ unity, by drawing a line from P through 1 to equal a, we have a point in the spiral; and setting the arc A1 from 1 to 2, and making a line from P through $2 = \frac{a}{2}$, we have another point in the spiral; and in like manner, by setting the arc 1, 2 from 2 to 3, and drawing a line from P through $3 = \frac{a}{3}$, we have another point in the spiral; and so on, indefinitely. Hence, drawing a curve with a steady hand through the points found, we shall have an approximate representation of the spiral, which evidently has EF for its asymptote.

REMARK.—It is manifest that this curve took its name from the striking analogy between its equation $r\theta = a$, and that of the hyperbola $xy = a^2$; see page 119 of Young's "Differential Calculus."

4. To find the subtangent in the spiral of Archimedes, whose equation is $r = a\theta$.

By taking the differentials of the members of its equation, we have $\frac{dr}{d\theta} = a =$ its subnormal $=$ const., and of course $r^2 \div \frac{dr}{d\theta} = r \times \frac{rd\theta}{dr} = r^2 \frac{1}{a} = r\theta$; consequently, the subtangent equals the length of a circular arc radius r, and angle that between r and the angular axis; see Young, page 118.

REMARK.—The equations of this and the hyperbolic spiral are included in the class of spirals represented by the equation $r = a\theta^n$; noticing, that n may be positive or negative, according to the nature of the case.

5. To find the subnormal and subtangent in the spiral, whose equation is $(r^2 - a^2)\,\theta^2 = b^2$.

Solving the equation with reference to r, we have

$$r^2 = a^2 + \frac{b^2}{\theta^2} \quad \text{or} \quad r = \sqrt{\left(a^2 + \frac{b^2}{\theta^2}\right)};$$

which gives

$$\frac{dr}{d\theta} = -\frac{b^2}{\theta^3\sqrt{a^2 + \frac{b^2}{\theta^2}}} = -\frac{b^2}{\theta^2\sqrt{a^2\theta^2 + b^2}} = \text{the subnormal.}$$

Dividing r^2 by the subnormal, we have $\frac{r^2 d\theta}{dr} = -\frac{(a^2\theta^2 + b^2)^{\frac{3}{2}}}{b^2}$ for the subtangent, which reduces to $-b$ when $\theta = 0$.

REMARKS.—1. From $r = \sqrt{\left(a^2 + \frac{b^2}{\theta^2}\right)}$, it is evident that the last value of r is a; which immediately follows from the proposed equation, when put under the form $\theta^2 = \frac{b^2}{r^2 - a^2}$.

Hence, if from the pole of the spiral as center with a as a radius, a circle is described, it will clearly be an asymptote to the spiral; since, when $r = a$, θ must be unlimitedly great, or must include an unlimitedly great number of circumferences.

2. In much the same way it may be shown that the spiral whose equation is $r = \sqrt{\left(a^2 - \frac{b^2}{\theta^2}\right)}$, lies wholly within the circumference of the circle whose radius is a; the circumference being an asymptote to the spiral.

6. To find the subnormal and subtangent of the spiral whose equation is $(r^2 - ar)\,\theta^2 = 1$.

From the equation we readily get $r^2 - ar = \frac{1}{\theta^2}$, and thence $r = \frac{a}{2} + \sqrt{\left(\frac{a^2}{4} + \frac{1}{\theta^2}\right)}$, which gives

$$\frac{dr}{d\theta} = -\frac{1}{\theta^3} \div \sqrt{\frac{a^2}{4} + \frac{1}{\theta^2}} = -\frac{2}{\theta^2\sqrt{a^2\theta^2 + 4}} = \text{the subnormal};$$

consequently, dividing r^2 by this, we readily get the subtangent.

REMARKS.—From $r = \frac{a}{2} + \sqrt{\left(\frac{a^2}{4} + \frac{1}{\theta^2}\right)}$, or the proposed equation, it follows that the circumference of the circle whose radius $= a$, is an asymptote to the spiral, being an interior asymptote; while the circle whose radius is a, is an exterior asymptote to the spiral whose equation is

$$(ar - r^2)\,\theta^2 = 1, \quad \text{or} \quad r = \frac{a}{2} + \sqrt{\left(\frac{a^2}{4} - \frac{1}{\theta^2}\right)},$$

the spiral falling wholly within the circle: see Young's "Differential Calculus," p. 123.

7. To find the subnormal and subtangent of the *parabolic spiral*, whose equation is $r^2 = a^2\theta$ or $r = a\theta^{\frac{1}{2}}$.

By taking the differentials, we have

$$\frac{dr}{d\theta} = \frac{1}{2}\frac{a}{\theta^{\frac{1}{2}}} = \frac{a}{2\sqrt{\theta}} = \text{the subnormal};$$

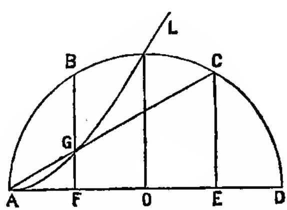
consequently, r^2 divided by the subnormal, gives

$$\frac{r^2 d\theta}{dr} = 2a\theta^{\frac{3}{2}} = \text{the subtangent},$$

and since, from the proposed equation $\theta^{\frac{1}{2}} = \frac{r}{a}$, we have $\frac{r^2 d\theta}{dr} = \frac{2r^3}{a^2}$.

REMARK.—It is manifest that the spiral is called parabolic, from the analogy of its equation $r^2 = a^2\theta$ to that of the parabola $y^2 = ax$.

8. To find the subtangent and subnormal at any point of the cissoid of Diocles.



Let ABCD be a semicircle, having AD for its diameter, to which the perpendicular ordinates BF and CE are drawn, at equal distances OF and OE from the center; then, drawing a right line from A to C, the extremity of the ordinate CE, it will intersect the other ordinate at a point G of the cissoid.

Representing AF, FG, and AD, severally, by x, y, and a; the equiangular triangles AFG and AEC will (from known principles of geometry) give the equal ratios expressed by $\frac{y}{x} = \frac{\text{CE}}{\text{AE}} =$ (by construction and the nature of the circle) $\frac{\text{BF}}{\text{DF}}$, or $\frac{y^2}{x^2} = \frac{\text{BF}^2}{\text{DF}^2} = \frac{\text{AF} \times \text{DF}}{\text{DF}^2} = \frac{\text{AF}}{\text{DF}} = \frac{x}{a-x}$ by the nature of the circle; consequently, we shall have $y^2 = \frac{x^3}{a-x}$, for the equation of the cissoid. By taking the differentials of the members of this equation, we have

$$2ydy = \frac{3x^2dx}{a-x} + \frac{x^3dx}{(a-x)^2} = \frac{x^2\,(3a-2x)\,dx}{2\,(a-x)^2},$$

which gives $\quad \frac{ydy}{dx} = \frac{x^2\,(3a-2x)}{2\,(a-x)^2} =$ the subnormal,

and dividing y^2 by the subnormal, we have

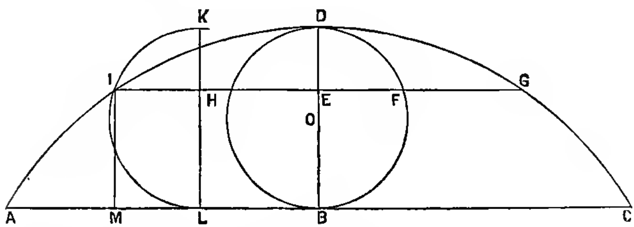
$$\frac{2x\,(a-x)}{3a-2x} = \text{the subtangent.}$$

If in $y^2 = \frac{x^3}{a-x}$ we put $x = a$, we shall have $y^2 = \frac{a^3}{a-a} = \frac{a^3}{0}$; consequently, y^2 is infinitely great, and of course there must be two infinite values of y, one of which is expressed by $+y$ and the other by $-y$; which must be asymptotes to the cissoid.

Thus, from $y^2 = \frac{x^3}{a-x}$ we have $y = \pm x\sqrt{\frac{x}{a-x}}$; consequently, any positive value of y being expressed by $y = x\sqrt{\frac{x}{a-x}}$, the corresponding negative value must be expressed by $y = -x\sqrt{\frac{x}{a-x}} = -\text{FG}$; of course, the lower part of the curve must be identical with the upper, being described in the lower semicircle, after the method that has been used in describing AGL. It is hence manifest that a perpendicular to the diameter AD through D, when produced infinitely both ways, will represent the asymptotes of the branches of the curve.

REMARK.—It is easy to perceive, that the upper and lower branches of the curve will touch each other at A, and will form with each other what is called *a cusp of the first kind*, since their convexities touch each other. It may be added, that if two branches of a curve touch each other in such a way that the convexity of one is in contact with the concavity of the other; then they form, at their point of contact, what is called *a cusp of the second kind.*

9. To draw a tangent and normal at any point of the common cycloid.



Let BFD represent the circumference of a circle having O for its center, OB $= r$ for its radius, DE and EF for the rectangular co-ordinates of the extremity F of the arc DF, the point D of the extremity of the diameter DB being taken for the origin of the co-ordinates; then, if the ordinate EF in the circle is produced to G so as to make FG = the circular arc DF, G will be a point in the cycloid. Hence, representing DE by x, and EG by y, we shall have

$$y = \text{EF} + \text{the arc DE},$$

or since DE $= x =$ the versed sine of the arc DF, and EF $=$ the sine of the same arc, when r, the radius of the circle, is taken for the radius; then, denoting the arc (according to usage) by ver $\sin^{-1} x$, we shall have

$$y = \text{ver} \sin^{-1} x + \sin \text{ver} \sin^{-1} x$$

for the equation of the cycloid, when the origin of the co-ordinates is taken at D, called *the vertex of the curve.* By taking the differentials of the members of the equation, we shall have

$$dy = \frac{rdx}{\sqrt{2rx - x^2}} + \frac{rdx - xdx}{\sqrt{2rx - x^2}} = \sqrt{\frac{2r - x}{x}}\, dx;$$

since (see page 73) $\dfrac{rdx}{\sqrt{(2rx - x^2)}}$ is the differential of the arc whose versed sine is x and radius r, and that $\dfrac{rdx - xdx}{\sqrt{2rx - x^2}}$ is the differential of the sine of the same arc. Hence,

$$\frac{dx}{dy} = \sqrt{\frac{x}{2r - x}} = \frac{x}{\sqrt{2rx - x^2}} = \frac{\text{DE}}{\text{EF}};$$

consequently, since $\dfrac{dx}{dy}$ equals the tangent of the angle which the tangent to the cycloid at G makes with EG, and that $\dfrac{\text{DE}}{\text{EF}}$ equals the tangent of the angle which the chord of the arc DF makes with EF, it results that the tangent to the cycloid at G is parallel to the chord DF of the corresponding arc DF of the circle. Hence a right line drawn through G parallel to the chord DF will be a tangent to the cycloid at G; and because the chords of the arcs DF and BF cut each other perpendicularly, it follows that a right line drawn through G parallel to the chord BF, and extended to meet DB produced toward B, will be a normal to the curve at G.

REMARKS.—1. If in $\frac{ydx}{dy}$ = the subtangent, we put for y its value, we shall have the subtangent =

$$\sqrt{\frac{x}{2r-x}} \times (\text{ver}\sin^{-1}x + \sin\text{ver}\sin^{-1}x);$$

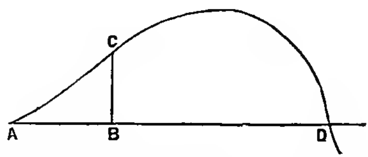
consequently, having computed the value of this, and set it off from E on BD produced toward D, by joining the extremity of the produced part with G, we shall clearly have the tangent, as derived from the subtangent and the point of contact.

2. Admitting the construction of the figure, and that KL, the diameter of the semicircle KIL perpendicular to AB, is equal to DB; we shall have AB = BC = the semicircumference DFB or KIL, arc KI = arc DF = FG = HE = LB (since EF = HI), and of course arc IL = AL. Hence, we shall have AM = AL − ML = arc IL − its sine IH; consequently, if y represents AM, and IM = HL = ver sin arc IL, we shall have $y = \text{ver}\sin^{-1}x - \sin\text{ver}\sin^{-1}x$ for another form of the equation of the cycloid; which may be regarded as being a transformation of the equation previously found, when the origin of the co-ordinates is changed without changing their directions.

3. The preceding equation clearly suggests the ordinary

method of describing the cycloid. Thus, conceiving the circle whose diameter is KL, to have the point I placed at A, and then rolled (without any sliding) from A toward C, the point I, in one revolution of the circle, will manifestly describe the cycloidal arc, ADC; noticing, that AC and BD are called the *base* and *axis* of the curve, and that the circle described on the axis is called the *generating circle.* It may be added, since x and $\sqrt{2rx - x^2}$ are (from the principles of trigonometry), not only the versed sine and sine of the arc IL, but of the arc IL increased or diminished by any number of times the circumference of the circle whose diameter is KL, we may suppose the circle to roll on infinitely along the right line AC produced to infinity toward C, and thereby to describe an unlimited number of successive cycloids, which will all be comprehended in the preceding equation.

10. To draw a tangent to the curve whose equation is $y = 3x + 18x^2 - 2x^3$.



By taking the differentials of the members of the equation, we have $\frac{dy}{dx} = 3 + 36x - 6x^2$; consequently, the equation of the tangent (see p. 126) $Y - y = \frac{dy}{dx}(X - x)$, is easily found for any proposed value of x.

Thus, if we put 1 for x, we have

$$\frac{dy}{dx} = 3 + 36 - 6 = 33, \quad \text{and} \quad y = 3 + 18 - 2 = 19;$$

consequently, the equation of the tangent is

$$Y - 19 = 33\,(X - 1), \quad \text{or} \quad Y = 33X - 14.$$

It is easy to perceive that this tangent cuts the curve; since, by putting x and y for X and Y, it is immediately reduced to $x^3 - 9x^2 + 15x - 7 = 0$, whose roots are $x = 1$, $x = 1$, and $x = 7$. The first two of these roots belong to the point of contact, while $x = 7$ is a point at which the tangent cuts the curve, having $y = 257$ for the corresponding ordinate of the curve.

If we put $x = 4$ in $\frac{dy}{dx} = 3 + 36x - 6x^2$, and proceed in the same way as before, we shall get $\frac{dy}{dx} = 51$, and thence the equation of the tangent to the curve at the point whose abscissa is 4, is $Y - 172 = 51\,(X - 4)$. Putting x and y for X and Y in this, we readily get $x^3 - 9x^2 + 24x - 16 = 0$; whose roots are $x = 4$, $x = 4$, and $x = 1$, the first two of which are the same as the abscissa of the proposed point; consequently, the tangent cuts the curve at the point whose abscissa $= 1$, and whose corresponding ordinate is $y = 19$.

Because the tangent to the curve at the point whose abscissa is 1 cuts the curve at the point whose abscissa is 7, while the tangent to the curve at the point whose abscissa is 4 cuts the curve at a point whose abscissa is 1, it is manifest that the first of these tangents must touch the convex part of the curve; that is, that part which is convex toward the axis of x; while the second tangent touches that part of the curve which is concave to the axis of x.

It is hence evident that there must be a point in the curve whose abscissa is between 1 and 4, such that the tangent to the curve will not cut the curve at any other point. Thus, the tangent to the curve at the point whose abscissa is 3, by putting 3 for x, reduces $\frac{dy}{dx} = 3 + 36x - 6x^2$ to $\frac{dy}{dx} = 57$, and $y = 3x + 18x^2 - 2x^3$ becomes $y = 117$; consequently, the equation of the tangent becomes $Y - 117 = 57\ (X - 3)$, or $Y = 57X - 54$. Hence, putting x and y for X and Y, we have $$3x + 18x^2 - 2x^3 - 117 = 57\ (x - 3),$$

or, $$3x + 18x^2 - 2x^3 = 57x - 54\ ,$$

which is equivalent to

$$x^3 - 9x^2 + 27x - 27 = 0;$$

whose roots are $x = 3$, $x = 3$, $x = 3$, and, of course, the tangent to the curve at the point whose abscissa $= 3$, cuts the curve at the same point. Because the curve changes the direction of its curvature at C, or at the point whose abscissa is 3, it is said to have *a point of inflection* or *contrary flexure* at C.

REMARKS.—1. It clearly results from what has been done, that in curves which suffer an inflection, a line which touches the curve on one side of a point of inflection may cut it on the other side.

2. Because the tangents cut the curve at their point of contact, it is clear that the points of contact of the tangents may be regarded as the union or coalescence of the two points in which the curve is cut by a secant, by regarding the points of intersection of the secant as being unlimitedly near each other. Also, because the tangent at the point of inflection does not cut the curve at any other point, it is clear that the tangent at this point ought to be regarded as being both a tangent and secant; that is, as cutting the curve and as tangent to its convex and concave arcs at the point, when taken separately.

3. If the point of contact of a tangent and two unlimitedly near points of intersection of a secant with a curve are supposed to be equivalent, it is easy to perceive that it results from what has been done, that the curve may be cut by a right line in three points, or as many as there are units in the degree of its equation. It is also manifest that curves which admit of a point of inflection must be at least of the third degree.

4. Supposing the equation of the tangent $Y-y=\frac{dy}{dx}(X-x)$ to be reduced to the form $\frac{Y-y}{X-x}-\frac{dy}{dx}=0$; then, if the tangent cuts the curve, or can be made (as in the question) to cut it at the point whose co-ordinates are X and Y, such that $\frac{Y-y}{X-x}$ may be regarded as a consecutive value of $\frac{dy}{dx}$, it is clear that for the preceding equation we may write $\frac{d^2y}{dx^2}=0$ or infinity, according to the nature of the case. See page 13.

Hence the points of inflection of a plane curve may be found by the following

RULE.

1. Let x and y represent the co-ordinates of a point of inflection, and suppose $\frac{dy}{dx}=\text{F}(x)=$ a function of x. Then, proceed, as in finding maxima and minima, to find the maxima and minima of $\text{F}(x)$; that is to say, find those roots of $\frac{d\text{F}(x)}{dx}=0$ which do not reduce $\frac{d^2\text{F}(x)}{dx^2}$ to naught, and they will correspond to points of inflection; if $\frac{d\text{F}(x)}{dx}=$ in-

finity, we must find the roots of $\frac{dx}{dF(x)} = 0$, and then, as in maxima and minima, find those which correspond to maxima or minima which will give points of inflection, while, if there are no maxima or minima, there can not be any points of inflection.

If the roots of $\frac{dF(x)}{dx} = 0$ are also roots of $\frac{d^2F(x)}{dx^2} = 0$, we must find the roots of $\frac{d^3F(x)}{dx^3} = 0$, provided they do not reduce $\frac{d^4F(x)}{dx^4}$ to naught; and so on, as in finding maxima and minima.

2. To determine for any value of x, whether the curve is convex or concave toward the axis of x, we substitute the value of x in $\frac{d^2y}{dx^2} = \frac{dF(x)}{dx}$; then, if the result has the same sign as y, it is easy to perceive that the convexity of the curve is turned toward the axis of x, and *vice versa*. Thus, from $\frac{dy}{dx} = 3 + 36x - 6x^2$, we have $\frac{dF(x)}{dx} = 36 - 12x$; which is clearly positive when x is less than 3, and the convexity of the curve is turned toward the axis of x; noticing, that $\frac{dF(x)}{dx} = 0$ gives $36 - 12x = 0$, or $x = 3$, and that $\frac{d^2F(x)}{dx^2} = -12$ shows $F(x)$ to be a maximum, when the curve passes from being convex toward the axis, to being concave.

To illustrate what has been done, take the following

EXAMPLES.

1. To find the point of inflection in the curve whose equation is $y = x^{\frac{1}{3}} + x^2$.

Here we have $\frac{dy}{dx} = \text{F}(x) = \frac{1}{2}x^{-\frac{1}{2}} + 2x$, which gives $\frac{d\text{F}(x)}{dx} = -\frac{1}{4}x^{-\frac{3}{2}} + 2$ and $\frac{d^2\text{F}(x)}{dx^2} = \frac{3}{8x^{\frac{5}{2}}}$; consequently, from $\frac{d\text{F}(x)}{dx} = 0$, we have $-\frac{1}{4}x^{-\frac{3}{2}} + 2 = 0$, or $x = \frac{1}{4}$; and since $\frac{d^2\text{F}(x)}{dx^2}$ is positive, F (x) is a minimum. And because $\frac{d\text{F}(x)}{dx} = -\frac{1}{4}x^{-\frac{3}{2}} + 2$ has a sign contrary to that of y when x is less than $\frac{1}{4}$, it is clear that its concavity before $x = \frac{1}{4}$ is turned toward the axis of x, but after $x = \frac{1}{4}$, the sign of $\frac{d\text{F}(x)}{dx}$ is the same as that of y; consequently the curve has an inflection at the point whose abscissa $= \frac{1}{4}$.

2. To find the point of inflection in the curve whose equation is $y^2 = x + x^3$.

From this equation we have

$$\frac{dy}{dx} = \text{F}(x) = \frac{1 + 3x^2}{2y},$$

and thence
$$\frac{d\text{F}(x)}{dx} = \frac{6x}{2y} - \frac{dy}{dx}\,\frac{1 + 3x^2}{2y^2} = 0,$$

or
$$6x = \frac{dy}{dx}\,\frac{1 + 3x^2}{y} = \frac{(1 + 3x^2)^2}{2(x + x^3)},$$

which readily reduces to $x^4 + 2x^2 = \frac{1}{3}$, whose solution gives

$$x \quad \left(\sqrt{\frac{4}{3}} - 1\right)^{\frac{1}{2}}.$$

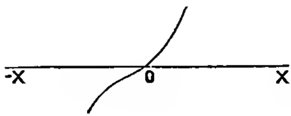
3. To find the point of inflection in the curve whose equation is $y = x^3$.

Here $\frac{dy}{dx} = \mathrm{F}(x) = 3x^2$, gives

$$\frac{d\mathrm{F}(x)}{dx} = 6x \quad \text{and} \quad \frac{d^2\mathrm{F}(x)}{dx^2} = 6\,;$$

consequently, from $\frac{d\mathrm{F}(x)}{dx} = 0$, we have $6x = 0$, or $x = 0$, and from $\frac{d^2\mathrm{F}(x)}{dx_2} = 6$, it follows that $x = 0$ makes $\mathrm{F}(x) = 3x^2$ a minimum.

Because $\frac{dy}{dx} = 3x^2$ equals 0 at the origin of the co-ordinates, it is clear that the curve has an inflection at the origin of the co-ordinates, where the curve touches the axis of x on the side of x positive and negative, so that the convexity of the curve above and below the axis of x is turned toward it. Thus the curve must be of the general form expressed by the adjoined figure.

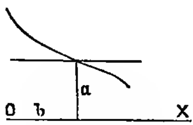


Remark.—Any curve whose equation is of the form $y = x^n$, such that n is an odd positive integer greater than one, must clearly be of the same general form as before.

4. To find the point of inflection of the curve whose equation is $y = a + (b - x)^{\frac{5}{3}}$.

Here $\frac{dy}{dx} = \mathrm{F}(x) = -\frac{5}{3}(b-x)^{\frac{2}{3}}$, gives $\frac{d\mathrm{F}(x)}{dx} = \frac{10}{9}(b-x)^{-\frac{1}{3}}$ or $\frac{dx}{d\mathrm{F}(x)} = \frac{9(b-x)^{\frac{1}{3}}}{10}$; consequently, putting this equal to naught, we have $x = b$, and a point of inflection may, of course, exist at this point. From $\frac{dy}{dx} = -\frac{5}{3}(b-x)^{\frac{2}{3}}$, for

$x = b$, we have $\frac{dy}{dx} = 0$, and, of course, the tangent to the curve at the extremity of the ordinate $y = a$ is parallel to the axis of x. From $\frac{dx}{dF(x)} = \frac{9(b-x)^{\frac{1}{3}}}{10}$ it is clear that when x is less than b, $\frac{dx}{dF(x)}$ will be positive, and when x is greater than b, $\frac{dx}{dF(x)}$ will be negative; consequently, the curve crosses the tangent at the extremity of the co-ordinate a, where it has a point of inflection.



Thus, as in the scheme, the curve passes through the extremity of the ordinate a, the point of inflection, and touches the tangent at the point, above and below, so that from the nature of a tangent its convexities will be turned toward the tangent.

5. To find the point of inflection in the curve whose equation is $y = mx + (b-x)^{\frac{5}{3}}$.

As in the preceding example, we take the differentials, and get

$$F(x) = \frac{dy}{dx} = m - \frac{5}{3}(b-x)^{\frac{2}{3}};$$

hence, as before,

$$\frac{dx}{dF(x)} = \frac{9}{10}(b-x)^{\frac{1}{3}},$$

and thence the curve has a point of inflection.

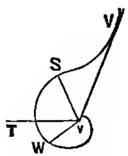
Hence clearly, if we change a in the preceding scheme into mb, and draw the tangent at its extremity to make an angle with the axis of x, whose tangent $= m$; then, the curve, when drawn with reference to the tangent, as in the scheme, will express the curve and its point of inflection, as required.

REMARK.—For the curve whose equation is

$$y = mx + (x - b)^{\frac{5}{3}},$$

we obtain the same results as before; with the exception, that the part of the curve toward the origin of the co-ordinates lies below the tangent, while the remaining part of it is above the tangent.

6. If, as is generally the case with spirals, the equation is referred to polar co-ordinates, then we may proceed to find its points of inflection as follows.



Thus, let WSV represent a spiral having r for its pole, rT for its angular axis, and rS for its radius vector, which makes the angle θ with rT; then, supposing the curve to have a point of inflection at S, it is manifest, since the tangent to the curve at S cuts it, and touches its convex and concave arcs at the same point, we may suppose the perpendicular from the pole r to the tangent at S to be constant, when a small change is made in the position of the point of contact; see Vince's "Fluxions," pp. 123 and 124.

Supposing with Vince, as we clearly may do, that the equation of the spiral is represented by $\theta = \left(\frac{r}{a}\right)^m$, by taking the differentials of its members, we have

$$d\theta = m\left(\frac{r}{a}\right)^{m-1}\frac{dr}{a}, \quad \text{or} \quad rd\theta = m\left(\frac{r}{a}\right)^{m} dr.$$

Hence, for simplicity, referring to the figure at p. 133, we have $rd\theta = \mathrm{S}c$ and $dr = b'c$, which give, since

$$(\mathrm{S}b')^2 = (\mathrm{S}c)^2 + (cb')^2,$$

$$\mathrm{S}b' = \sqrt{r^2\,d\theta^2 + dr^2} = \left\{1 + m^2\left(\frac{r}{a}\right)^{2m}\right\}^{\frac{1}{2}} dr.$$

Hence, drawing the perpendicular p from the pole r, to the tangent Tt, we get from similar triangles,

$$\mathrm{S}b' : \mathrm{S}c :: \mathrm{S}r : p = \frac{r^2 d\theta}{\mathrm{S}b'} = \frac{m\left(\frac{r}{a}\right)^m rdr}{\left\{1 + m^2\left(\frac{r}{a}\right)^{2m}\right\}^{\frac{1}{2}} dr} = \frac{mr^{m+1}}{(a^{2m} + m^2 r^{2m})^{\frac{1}{2}}};$$

which agrees with the perpendicular Sy found by Vince, at p. 124.

Because of the supposed constancy of p, the differential of this must be put equal to naught, and of course the differential of its square must also equal naught; consequently, we shall have

$$\frac{(2m + 2)\, m^2 r^{2m+1} dr}{a^{2m} + m^2 r^{2m}} - \frac{2m^5 r^{4m+1}}{(a^{2m} + m^2 r^{2m})^2} = 0,$$

or $$2m^4 r^{4m+1} + (2m + 2)\, m^2 a^{2m} r^{2m+1} = 0,$$

which readily reduces to $r = \left(-\frac{m+1}{m^2}\right)^{\frac{1}{2m}} \times a$; the same conclusion as obtained by Vince. To make r positive and real, it is clearly necessary that m should be negative and numerically greater than 1; thus, if $m = -2$, we have $r = \left(\frac{1}{4}\right)^{-\frac{1}{4}} a = (4)^{\frac{1}{4}} \times a = \sqrt{2} \times a$, and the equation of the spiral $\theta = \left(\frac{r}{a}\right)^m$ becomes $\theta = \left(\frac{r}{a}\right)^{-2}$ or $\frac{r}{a} = \theta^{-\frac{1}{2}}$, which is equivalent to $r = a\theta^{-\frac{1}{2}}$, the equation of the spiral that is called the *lituus*. If $m = -3$, the preceding equation gives $r = \left(\frac{9}{2}\right)^{\frac{1}{6}} \times a$, and the equation of the spiral is $\theta = \left(\frac{r}{a}\right)^{-3}$ or $r = a\theta^{-\frac{1}{3}}$; and in like manner the spirals whose equations are $r = a\theta^{-\frac{1}{4}}$, or $r = a\theta^{-\frac{1}{5}}$, and so on, have points of inflection; while all those spirals in which m is not negative and numerically greater than 1, have no points of inflection.

SECTION VI.

RADII OF CURVATURE, INVOLUTES AND EVOLUTES, ETC.

(1.) SUPPOSING $(x - x')^2 + (y - y')^2 = r^2$ to be the equation of a circle, whose radius is r and the rectangular co-ordinates of its center are x' and y'; then, if $y = \mathrm{F}(x)$ represents the equation of any plane curve, such that we can find $\frac{dy}{dx}$ and $d\,\frac{dy}{dx} \div dx$ from it, so that they shall be the same as in the circle; then, the radius of the circle is called the radius of curvature of the curve at the point, whose co-ordinates are expressed by x and y. Representing $\frac{dy}{dx}$ by p and $d\,\frac{dy}{dx} \div dx = \frac{dp}{dx}$ by p' in the proposed curve, by taking the first and second differentials of the equation of the circle, on the supposition of the constancy of r, x', and y', we shall have $(y - y')\,dy + (x - x')\,dx = 0$; or, since $\frac{dy}{dx}$ and $d\,\frac{dy}{dx} \div dx$ must equal p and p', we shall have $(y - y')\,p + x - x' = 0$, whose differential gives $(y - y')\,p' + p^2 + 1 = 0$. From $(y - y')\,p + x - x' = 0$ and $(x - x')^2 + (y - y')^2 = r^2$ we get $(y - y')^2\,(p^2 + 1) = r^2$, and from $(y - y')\,p' = -\,(p^2 + 1)$ we have $(y - y')^2 p'^2 = (p^2 + 1)^2$; consequently, from substitution we shall have $r^2 = \frac{(p^2 + 1)^3}{p'^2}$ or $r = \frac{(p^2 + 1)^{\frac{3}{2}}}{p'}$; by taking the sign of the right member of this equation such, that r may be positive.

(2.) There is another way of obtaining the preceding expression for r, which it may be proper to notice in this place.

Thus, from p. 129 we may clearly represent the normal at any point of the curve $y = \mathrm{F}(x)$, whose co-ordinates are y and x, by $\mathrm{Y} - y = -\dfrac{(\mathrm{X} - x)}{\frac{dy}{dx}}$, or its equivalent,

$$(y - \mathrm{Y})\,p + x - \mathrm{X} = 0,$$

which is the same as given above when for X and Y we put x' and y'; consequently, differentiating this on the supposition of the constancy of x' and y', we shall, as before, get $r = \dfrac{(p^2+1)^{\frac{3}{2}}}{p'}$ for the radius of curvature of the proposed curve. If the tangent at the proposed point of the curve is parallel to the axis of x, or the axis of y coincides in direction with the normal, we shall clearly have $p = 0$; and $r = \dfrac{1}{p'}$. If the curve has a point, such, that (without regard to p) $p' = d\,\dfrac{dy}{dx} \div dx = 0$, then, by taking x for the independent variable, we shall have $\dfrac{d^2y}{dx^2} = 0$, or infinity, which (since $r =$ infinity) clearly shows that the circumference becomes a right line at the point which touches and cuts the curve at the point; and of course the point is generally a point of *inflection*, agreeably to what has been shown.

To illustrate what has been done, take the following

EXAMPLES.

1. To find the radius of curvature at any point of the logarithmic curve whose equation is $y = a^x$, or, taking the hyperbolic logarithms, $\log y = x \log a$.

Taking the differentials, we have

$$\frac{dy}{y} = dx \log a \quad \text{or} \quad \frac{dy}{dx} = p = y \log a,$$

and $$\frac{dp}{dx} = p' = \frac{dy}{dx} \log a = p \log a.$$

Hence, representing $\log a$ by $\frac{1}{m}$, we shall have

$$p^2 + 1 = \frac{y^2}{m^2} + 1 = \frac{y^2 + m^2}{m^2} \quad \text{and} \quad p' = \frac{y}{m^2};$$

consequently, from $r = \frac{(p^2+1)^{\frac{3}{2}}}{p'}$, we shall have

$$r = \frac{(y^2 + m^2)^{\frac{3}{2}}}{m^3} \div \frac{y}{m^2} = \frac{(m^2 + y^2)^{\frac{3}{2}}}{my},$$

for the radius of curvature: noticing, that the center of the circle must (see fig. p. 142) be taken on the concave side of the curve.

2. To find the radius of curvature at any point of the common cycloid.

From p. 150 we have $\frac{dy}{dx} = p = \sqrt{\frac{2r - x}{x}}$,

and $$\frac{dp}{dx} = p' = \frac{-\frac{r}{x^2}}{\sqrt{\left(\frac{2r}{x} - 1\right)}} = \frac{-r}{x\sqrt{(2rx - x^2)}}.$$

Hence we shall have

$$p^2 + 1 = \frac{2r}{x} \quad \text{and} \quad (p^2 + 1)^{\frac{3}{2}} = \frac{2r}{x}\sqrt{\frac{2r}{x}};$$

consequently, $(p^2 + 1)^{\frac{3}{2}} \div p' = 2\sqrt{2r \times (2r - x)} =$ the radius of curvature. Thus it is manifest that the radius of curvature equals twice the corresponding normal of the cycloid; so that (see the fig. at p. 149) the radius of curvature at G equals twice the chord BF of the arc of the generating circle, which corresponds to the cycloidal arc GC.

3. To find the radius of curvature in the parabola, whose equation is $y^2 = 4mx$.

By taking the differentials we have $2ydy = 4mdx$, which gives $\frac{dy}{dx} = p = \frac{2m}{y}$, and from this $\frac{dp}{dx} = p' = -\frac{2mp}{y^2}$.

Hence $$p^2 + 1 = \frac{4m^2}{y^2} + 1 = \frac{4m^2 + y^2}{y^2}$$

gives $$(p^2 + 1)^{\frac{3}{2}} = \frac{(4m^2 + y^2)^{\frac{3}{2}}}{y^3},$$

and thence $r = \frac{(p^2 + 1)^{\frac{3}{2}}}{p'} = \frac{(4m^2 + y^2)^{\frac{3}{2}}}{-2myp}$, or taking it with the positive sign, we have $r = \frac{(4m^2 + y^2)^{\frac{3}{2}}}{2myp}$ for the radius of curvature; or, since $p = \frac{2m}{y}$, we have

$$r = \frac{(4m^2 + y^2)^{\frac{3}{2}}}{4m^2} = 2\,\frac{(m^2 + mx)^{\frac{3}{2}}}{m^2} = \frac{(\text{normal})^3}{4m^2}.$$

(See Young's "Differential Calculus," p. 131.)

4. To find the radius of curvature in the ellipse, whose equation is $a^2y^2 + b^2x^2 = a^2b^2$; a and b representing the half major and minor axes.

Differentiating, we have $a^2yp + b^2x = 0$, or $p = -\frac{b^2x}{a^2y}$, which gives

$$p' = -\frac{b^2}{a^2y} + \frac{b^2px}{a^2y^2} = -\frac{b^2 + a^2p^2}{a^2y} = -\frac{b^2(a^2y^2 + b^2x^2)}{a^4y^3} = -\frac{b^4}{a^2y^3}.$$

Hence, $p^2 + 1 = \frac{a^4y^2 + b^4x^2}{a^4y^2}$, and thence $(p^2 + 1)^{\frac{3}{2}} = \frac{(a^4y^2 + b^4x^2)^{\frac{3}{2}}}{a^6y^3}$;

consequently, the radius of curvature $r = \frac{(a^4y^2 + b^4x^2)^{\frac{3}{2}}}{a^4b^4}$:

and since $a^2y^2 - b^2x^2 = -a^2b^2$ is the equation of the hyperbola, its radius of curvature is evidently of the same form.

From what is shown on p. 128, $\frac{ydy}{dx} = py =$ the subnormal, and from $a^2py + b^2x = 0$ we have $py = -\frac{b^2x}{a^2} =$ the subnormal; and thence, from what is shown at the same place, we have

$$\sqrt{(y^2+p^2y^2)} = \text{the normal} = \text{N} = \left(y^2 + \frac{b^4}{a^4}x^2\right)^{\frac{1}{2}} = \frac{(a^4y^2+b^4x^2)^{\frac{1}{2}}}{a^2};$$

consequently, from substitution, we have $r = \frac{a^2\text{N}^3}{b^4}$: noticing, since $y^2 = \frac{b^2}{a^2}(a^2 - x^2)$, that N may be written in the form

$$\left(\frac{b^4}{a^4}x^2 + \frac{b^2}{a^2}(a^2-x^2)\right)^{\frac{1}{2}} = \frac{b}{a}\left\{a^2 - \left(1-\frac{b^2}{a^2}\right)x^2\right\}^{\frac{1}{2}} = \frac{b}{a}\left(a^2 - \frac{a^2-b^2}{a^2}x^2\right)^{\frac{1}{2}},$$

or, according to custom, representing $\frac{a^2-b^2}{a^2}$ by e^2, we shall have $\text{N} = \frac{b}{a}(a^2-e^2x^2)^{\frac{1}{2}}$.

Substituting this value of N in $r = \frac{a^2\text{N}^3}{b^4}$, we shall have $r = \frac{(a^2-e^2x^2)^{\frac{3}{2}}}{ab}$, in which $(a^2-e^2x^2)^{\frac{1}{2}} =$ the semidiameter of the ellipse, parallel to the tangent passing through the point of contact of the circle. See Young, p. 132.

Because $x^2 = \frac{a^2}{b^2}(b^2 - y^2)$, we readily get

$$\text{N} = \frac{\sqrt{[b^4 + (a^2-b^2)\,y^2]}}{a} = \frac{\sqrt{(b^4 + a^2e^2y^2)}}{a};$$

consequently, if N makes the angle L with the major axis of the ellipse, we shall clearly have $y = \text{N} \sin \text{L}$, and thence

$$N = \frac{\sqrt{(b^4 + a^2e^2 N^2 \sin^2 L)}}{a},$$

which gives $$N = \frac{b^2}{a} \div (1 - e^2 \sin^2 L)^{\frac{1}{2}}.$$

Substituting this value of N in r, we shall readily get

$$r = \frac{b^2}{a\ (1 - e^2\ \sin^2 L)^{\frac{3}{2}}} = \frac{a\ (1 - e^2)}{(1 - e^2 \sin^2 L)^{\frac{3}{2}}},$$

a formula that is very useful in determining the figure of the earth, L being the apparent latitude of the point of contact of the circle with the ellipse. See Young, pp. 132 and 133.

REMARKS.—1. The radius of curvature $r = \frac{(p^2+1)^{\frac{3}{2}}}{p'}$ may be put under several different forms, which are often used. Thus, since $p = \frac{dy}{dx}$, we have $p^2+1 = \frac{dy^2+dx^2}{dx^2}$, which gives $(p^2+1)^{\frac{3}{2}} = \frac{(dy^2 + dx^2)^{\frac{3}{2}}}{dx^3}$, and from $p' = d\,\frac{dy}{dx} \div dx$, this becomes, after dividing by p', $\frac{(p^2 + 1)^{\frac{3}{2}}}{p'} = \frac{(dy^2 + dx^2)^{\frac{3}{2}}}{dx^2 d\,\frac{dy}{dx}}$.

From what is shown at pp. 125 and 126 (see fig. at p. 125), it is manifest, since $SQ = dx$ and $GR = dy$ are common to the tangent line and curve, that $SR = \sqrt{dy^2 + dx^2}$ must be the differential of the right line TS and the curve AS. Hence, if s represents an arc of a curve and ds its differential, we shall have $r = \frac{(dy^2 + dx^2)^{\frac{3}{2}}}{dx^2\, d\,\frac{dy}{dx}} = \frac{ds^3}{dx^2\, d\,\frac{dy}{dx}}$ for its radius of curvature; in which, without destroying its generality, we may for $d\,\frac{dy}{dx}$ take its differential on the supposition that either dx or dy is constant, or x or y taken for the independent variable.

Thus, since $$d\frac{dy}{dx} = \frac{d^2y dx - d^2x dy}{dx^2},$$

we shall have $$r = \frac{ds^3}{d^2y dx - d^2x dy};$$

which, by taking x for the independent variable, reduces to $r = \frac{ds^3}{d^2y dx}$, since $dx =$ const. gives $d^2x = 0$; and if y is taken for the independent variable, it becomes $r = \frac{ds^3}{-d^2x dy}$. If we take ds for the independent variable, we shall have $dy^2 + dx^2 =$ const., and, of course, its differential gives

$$dy d^2y + dx d^2x = 0; \quad \text{which gives} \quad d^2x = -\frac{dy d^2y}{dx},$$

or $$d^2y = -\frac{dx d^2x}{dy}.$$

From the substitution of these values in $d^2y dx - d^2x dy$, we have $d^2y dx - d^2x dy = \frac{dy^2 + dx^2}{dx} \times d^2y = \frac{ds^2 d^2y}{dx}$, and $d^2y dx - d^2x dy = -\frac{ds^2 d^2x}{dy}$; consequently, from the substitution of these values in $r = \frac{ds^3}{d^2y dx - d^2x dy}$, we shall have

$$r = \frac{dx ds}{d^2y}, \quad \text{and} \quad r = -\frac{dy ds}{d^2x}.$$

2. By referring to the figure given at p. 125, it is manifest from the nature of the right line, that if we pass along the tangent and assume SQ to be constant, RQ $= dy$ will also be constant, while, if we pass along the curve concave to the axis of x and suppose dx to be constant, RQ $= dy$ will decrease, and, of course, d^2y or $\frac{d^2y}{dx^2}$ must be negative; and it

is clear that d^2y and $\frac{d^2y}{dx^2}$ must be positive, when the convexity of the curve is turned toward the axis of x.

Hence, because the radius of curvature must always be positive, it is clear that in applying $r = \frac{(p^2+1)^{\frac{3}{2}}}{p'}$ and the preceding derived formulas to practice, p' and d^2y must be taken with the negative sign in them when the curve is concave toward the axis of x, and with the positive sign when the convexity is turned toward the axis of x.

(3.) Resuming the equation $y = \mathrm{F}(x)$, and the equations $(y - y')\,p + x - x' = 0$, $(y - y')\,p' = -(p^2 + 1)$, from page 163, it is manifest that if we find x and y from any two of these in terms of x' and y' and known terms, that by substituting them in the third equation we shall have an equation between x' and y', which will be the equation of the curve in which the centers of all the radii of curvature of the proposed curve must lie.

Where it is to be noticed, that the equation $y = \mathrm{F}(x)$ is called the *involute of the curve thus found;* which is called the *evolute of* $y = \mathrm{F}(x)$, *or of the involute.*

The reason for these denominations is plain, from the circumstance that we may regard the involute as being generated by the unlapping of a thread placed in contact with the evolute, in such a way that the part unlapped at any point equals the corresponding radius of curvature, when its extremity will be in a point of the involute. Where it is manifest, that the radius of curvature is always a tangent to the evolute, and constantly perpendicular to a tangent to the involute at its extremity.

For convenience in practice, we may give the last two of the preceding equations the forms

$$y' = y + \frac{p^2+1}{p'} \quad \text{and} \quad x' = x - \frac{p(p^2+1)}{p'},$$

which may be freed from $p = \frac{dy}{dx}$ and $p' = d\frac{dy}{dx} \div dx$, by finding their values from $y = F(x)$, the equation of the involute, or that of the proposed curve; when we may proceed as directed above.

To illustrate what has been done, take the following

EXAMPLES.

1. To find the evolute of the parabola, whose equation is $y^2 = 4mx$.

Here for $y = F(x)$, we have $y^2 = 4mx$; which gives

$$p = \frac{dy}{dx} = \frac{2m}{y} \quad \text{and} \quad p' = d\frac{dy}{dx} \div dx = -\frac{2pm}{y^2} = -\frac{4m^2}{y^3}.$$

Hence we have $p^2 + 1 = \frac{y^2+4m^2}{y^2}$, and thence

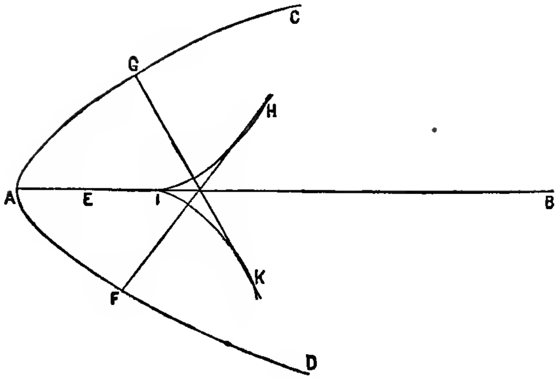
$$y' = y - \frac{y^3}{4m^2} - y = -\frac{xy}{m} \quad \text{or} \quad x = \left(\frac{my'^2}{4}\right)^{\frac{1}{3}}$$

and $\quad y = -(4m^2y')^{\frac{1}{3}}$. We also have

$$x' = x + \left(\frac{y^3}{4m^2} + y\right)\frac{2m}{y} = 3x + 2m \quad \text{or} \quad x = \frac{x'-2m}{3};$$

consequently, by equating these values of x, we readily get $y'^2 = \frac{4}{27}(x'-2m)^3 \div m$ for the equation of the evolute, which is of the form of the well-known equation of the *semicubical parabola* If the origin of the co-ordinates is moved in the direction of x positive through the distance $2m$, or that we put $x' - 2m = x$ and use y for y', the equation of the evolute may be more simply expressed by the form $y^2 = \frac{4x^3}{27m}$.

Thus, let CAD represent a parabola having AB for its axis, A for its vertex, and E for its focus; then, by setting



off EI in the direction of x positive equal to AE, it is clear that we shall have $AI = 2m$ for the radius of curvature of the parabola at its vertex, which equals $4m \div 2$, or half its principal parameter, or latus rectum. Then, since $y^2 = \frac{4x^3}{27m}$ gives $y = \pm \sqrt{\frac{4x^3}{27m}}$, we construct the curve HIK, having I for its vertex by setting off $y = \sqrt{\frac{4x^3}{27m}}$ at the distance x above IB, and $y = -\sqrt{\frac{4x^3}{27m}}$ at the distance x below IB; consequently, a curve passing through all the points thus found, will represent the evolute of the parabola, or the semi-cubical parabola.

Supposing the evolute to be correctly constructed, then a thread stretched from A to I, and lapped on the branch IK so as to coincide with it, and made fast at its unlimitedly remote extremity, when unlapped, by moving the extremity A toward C and keeping it stretched, the point A will clearly describe that part of the parabola represented by AC. By lapping the thread on IH instead of IK, we may in like manner describe the branch of the parabola represented by AD.

Remark.—Mr. Young, at page 140 of his "Differential Calculus," says that the evolute does not extend on the side of x negative, or from I toward A, since x negative in $y^2 = \frac{4x^3}{27m}$ will make y imaginary, which is undoubtedly true; yet from the first form $y'^2 = \frac{4(x' - 2m)^3}{27m}$ of the evolute, which, for $x = 0$ in the second form, gives $x' = 2m$, clearly shows that the point A is so connected with the evolute, that AI must be taken in conjunction with it, as has been done in the preceding construction; and it is manifest that like observations will be applicable in all analogous cases.

2. To find the evolute of the common cycloid.

From Ex. 2, at p. 165, we have

$$\frac{dy}{dx} = p = \sqrt{\frac{2r - x}{x}}, \quad p^2 + 1 = \frac{2r}{x},$$

and
$$p' = -\frac{r}{x\sqrt{(2rx - x^2)}};$$

hence, $y' = y + \frac{p^2 + 1}{p'}$ and $x' = x - \frac{p(p^2 + 1)}{p'}$,

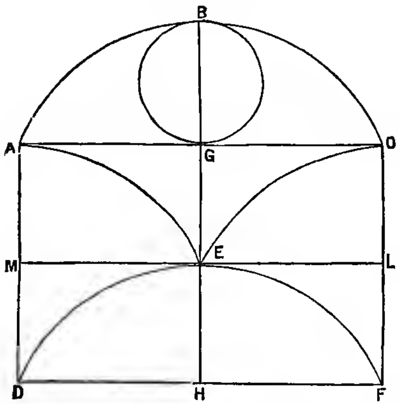
from p. 171, will, by substitution, become

$$y' = y - 2\sqrt{2rx - x^2} \quad \text{and} \quad x' = x + 4r - 2x = 4r - x.$$

Hence, from the substitution of the value of y, from p. 150, in that of y', we shall have

$$y = \text{ver sin}^{-1}x + \text{sin ver sin}^{-1}x - 2\sqrt{(2rx - x^2)}$$
$$= \text{ver sin}^{-1}x - \sqrt{(2rx - x^2)},$$

or $y' = \text{ver sin}^{-1}x - \text{sin ver sin}^{-1}x$; which, from what is shown at p. 151, is the equation of a cycloid, the origin of the co-ordinates being at the extremity of its base. From $x' = 4r - x$ we have $x = 4r - x'$, from which it is manifest (see fig. at p. 149), since $DB = 2r$ and that D is the origin of the co-ordinates, that if DB is produced below B to the distance DB or $2r$, and then x' subtracted from $4r$, we shall readily get $-x = x' - 4r$; which clearly shows that we may change the origin of the co-ordinates from D to the point distant $2r$ below B, and reckon the positive values of x from the new origin upward instead of downward, when the origin is at D.



Hence, supposing ABC to represent the proposed cycloid, by removing the origin of the co-ordinates from the vertex B to E, so that GE = GB, we may reckon x positive from E toward G, and $y' = \text{ver sin}^{-1} x - \sin \text{ver sin}^{-1} x$, the equation of the semicycloid EC, whose semibase is EL and vertex C, will be that of the evolute of the proposed semicycloid BC; and in like manner the semicycloid AE, whose semibase is EM and vertex A, is the evolute of the semicycloid BA:

the proposed semicycloids and their evolutes being clearly identical.

REMARKS.—1. The cycloid DEF being drawn (as in the figure) equal to the proposed cycloid ABC, it is evident that the semicycloid EF and ED will be evolutes of EC and EA; and so on, indefinitely, for semicycloids that may be described below the cycloid DEF, like EC and EA below the cycloid ABC. And it is easy to perceive that a series of cycloids may in this way be continued indefinitely, both above and below the proposed cycloid ABC.

2. To describe the involutes by the evolutes, we take a thread equal to the semicycloidal arc EC, and fasten one of its extremities at E; then, having lapped it on the arc EC, we carry the extremity C from C through B to A, when the cycloidal arc CBA will evidently have been described.

To describe the arcs EC and EA, we use two threads tied to the points D and F, equal in length to the arcs DE and FE; then, the extremities at E, being carried from E to A and C, will describe the semicycloidal arcs AE and EC.

3. To find the equation of the evolute of the ellipse.

From p. 166, we have $a^2y^2 + b^2x^2 = a^2b^2$ for the equation of an ellipse, and

$$p = -\frac{b^2x}{a^2y}, \quad \text{and} \quad p' = -\frac{b^4}{a^2y^3},$$

and

$$p^2 + 1 = \frac{a^4y^2 + b^4x^2}{a^4y^2}, \quad \text{and} \quad \frac{p}{p'} = \frac{xy^2}{b^2}.$$

Hence, from p. 171, the equations

$$y' = y + \frac{p^2+1}{p'} \quad \text{and} \quad x' = x - \frac{p(p^2+1)}{p'},$$

become

$$y' = y - \frac{y(a^4y^2 + b^4x^2)}{a^2b^4} \quad \text{and} \quad x' = x - \frac{x(a^4y^2 + b^4x^2)}{a^4b^2}.$$

From the equation of the ellipse we have $a^2y^2 = a^2b^2 - b^2x^2$, which, substituted for a^2y^2 in the value of y', reduces it to $y' = y - \frac{y\,[a^4b^2 - b^2(a^2 - b^2)\,x^2]}{a^2b^4}$, which, putting $a^2 - b^2 = c^2$, is easily reduced to

$$y' = y - \frac{y\,(a^4 - c^2x^2)}{a^2b^2} = -\frac{c^2(a^2 - x^2)\,y}{a^2b^2} = -\frac{c^2y^3}{b^4};$$

and, in a similar way, we have $x' = \frac{c^2x^3}{a^4}$.

Hence, we readily get

$$y^6 = \frac{b^8y'^2}{c^4} \quad \text{or} \quad y^2 = \left(\frac{b^8y'^2}{c^4}\right)^{\frac{1}{3}}, \quad \text{and} \quad x^2 = \left(\frac{a^8x'^2}{c^4}\right)^{\frac{1}{3}};$$

consequently, from the substitution of these values of y^2 and x^2 in $a^2y^2 + b^2x^2 = a^2b^2$, we have $a^2\left(\frac{b^8y'^2}{c^4}\right)^{\frac{1}{3}} + b^2\left(\frac{a^8x'^2}{c^4}\right)^{\frac{1}{3}} = a^2b^2$

or $$(b^2y'^2)^{\frac{1}{3}} + (a^2x'^2)^{\frac{1}{3}} = (by')^{\frac{2}{3}} + (ax')^{\frac{2}{3}} = c^{\frac{4}{3}},$$

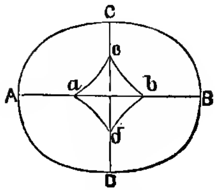
and of course $(by')^{\frac{2}{3}} + (ax')^{\frac{2}{3}} = (a^2b^2)^{\frac{2}{3}}$ is the equation of the evolute of the ellipse.

By putting $x' = 0$, the equation reduces to

$$(by')^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}},$$

or its equivalent $by' = a^2 - b^2$, which gives $y' = \frac{a^2}{b} - b$; and in like manner, by putting $y' = 0$, the same equation gives $x' = a - \frac{b^2}{a}$.

Thus, let AB $= 2a$ and CD $= 2b$ be the major and minor axes of the ellipse, and let the points c and d be taken on the minor axis at the distances $\frac{a^2}{b} - b$ from the center, while a and b are taken on the major axis at distances equal to $a - \frac{b^2}{a}$ from the center; then, curves drawn, as in the figure,



with their convexities toward the axes, so as to touch them at their extremities, will represent the evolutes of the ellipse. It is manifest that $\frac{a^2}{b}$ and $\frac{b^2}{a}$ are the radii of curvature of the ellipse at the extremities of the minor and major axes.

The ellipse may be described by means of its evolute as follows:

Take a thread, in length equal to the arc $cb + \text{B}b$, and fasten one end of it at c, and lap it on the arc cb, and bring down the remaining part of it to B; then, carry the thread around from B to A, and its extremity B will describe the half of the ellipse represented by BDA; and it is manifest that having fastened the extremity of the thread at d, we may in like manner describe the remaining half of the curve, represented by BCA.

Because the arc $cb + \text{B}b =$ the arc $cb + \frac{b^2}{a} = c\text{D} = \frac{a^2}{b}$, it follows that the arc $cb = \frac{a^2}{b} - \frac{b^2}{a} = \frac{a^3 - b^3}{ab}$; consequently, since the four branches of the evolute are clearly equal to each other, we shall have $4\frac{a^3 - b^3}{ab}$ for the entire length of the evolutes. Hence, if b is very small in comparison to a, it is clear that the arc cb will differ but little from $\frac{a^2}{b}$; consequently, the points c and d will fall ultimately without the ellipse, and the semi-ellipses BDA and BCA will have for their limits arcs of circles whose centers are at c and d, and are drawn through the points B, D, A and B, C, A.

REMARKS.—1. Resuming the equation

$$(b^2y'^2)^{\frac{1}{3}} + (a^2x'^2)^{\frac{1}{3}} = (a^2b^2)^{\frac{2}{3}}$$

of the evolute from p. 176, then, siuce the equation

$$a^2y^2 + b^2x^2 = a^2b^2$$

becomes $$a^2y^2 - b^2x^2 = -a^2b^2,$$

the equation of an hyperbola, by changing b^2 into $-b^2$, it clearly follows that if for b^2 we put $-b^2$ in the preceding equation of the evolute, it will be reduced to

$$-(b^2y'^2)^{\frac{1}{3}} + (a^2x'^2)^{\frac{1}{3}} = (a^2 + b^2)^{\frac{2}{3}},$$

or its equivalent $(ax')^{\frac{2}{3}} = (a^2 + b^2)^{\frac{2}{3}} + (by')^{\frac{2}{3}}$, the equation of the evolute of the hyperbola. If we put $y' = 0$, the equation reduces to

$$ax' = a^2 + b^2, \quad \text{or} \quad x' = a + \frac{b^2}{a},$$

which clearly shows that $\frac{b^2}{a}$ equals the radius of curvature at the extremity of the major axis ($2a$) of the hyperbola.

By assuming y', we can, from the above equation, evidently find the corresponding value of x'; and in this way find any number of points, at pleasure, of the evolute.

2. It is easy to perceive that we can not, from the preceding equation, find the evolute of the conjugate hyperbolas; which clearly shows that their evolute is different from that which has been found.

To find the evolute of the conjugate hyperbolas, we must proceed in much the same way as before, by regarding b as their principal semi-axis, and a as its semiconjugate; consequently, we shall, as before, have $(by')^{\frac{2}{3}} = (a^2 + b^2)^{\frac{2}{3}} + (ax')^{\frac{2}{3}}$ for the equation of the evolute of either of the conjugate hyperbolas. By putting $x' = 0$ in this equation we readily get $by' = a^2 + b^2$, which gives $y' = b + \frac{a^2}{b}$, and shows that $\frac{a^2}{b}$ is the radius of curvature at the vertex of either of the conjugate hyperbolas. By assuming x' we can, from the preceding equation, calculate y', and thence find, at will, any number of points in the evolute.

(4.) We will now proceed to show how to find the radii of curvature of curves, whose equations are expressed in polar co-ordinates.

Supposing, as at p. 131, that $r \cos \omega = -r \cos \theta = x$ and $r \sin \omega = r \sin \theta = y$, by taking the differentials of

$$x = -r \cos \theta \quad \text{and} \quad y = r \sin \theta,$$

on the supposition that θ is the independent variable, we shall have

$$dx = -\cos \theta\, dr + r \sin \theta d\theta,\ dy = \sin \theta dr + r \cos \theta d\theta,$$

whose squares added give $dx^2 + dy^2 = dr^2 + r^2 d\theta^2$.

And from $d\,\frac{dy}{dx} = d\left(\frac{\sin \theta dr + r \cos \theta d\theta}{-\cos \theta dr + r \sin \theta d\theta}\right)$

we readily get

$$d^2y dx - d^2x dy = -r^2 d\theta^3 - 2dr^2 d\theta + r d^2 r d\theta;$$

consequently, since (see p. 168) the radius of curvature, r',

equals $$\frac{(dy^2 + dx^2)^{\frac{3}{2}}}{dx^2 d\,\frac{dy}{dx}} = \frac{(dr^2 + r^2 d\theta^2)^{\frac{3}{2}}}{d^2y dx - d^2x dy},$$

we shall of course have for r' the expression

$$\frac{(dr^2 + r^2 d\theta^2)^{\frac{3}{2}}}{(-r^2 d\theta^2 - 2dr^2 + r d^2 r)\, d\theta};$$

noticing, that the expression can be put under the more simple form,

$$r' = \left\{\left(\frac{dr}{d\theta}\right)^2 + r^2\right\}^{\frac{3}{2}} \div \left\{-r^2 - 2\left(\frac{dr}{d\theta}\right)^2 + r\,\frac{d^2r}{d\theta^2}\right\}$$

which must be taken with the positive sign.

If N represents the *polar normal*, since

$$N = \sqrt{(dr^2 + r^2d\theta^2)} \div d\theta,$$

we readily get $$r' = \frac{-N^3}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\,\frac{d^2r}{d\theta^2}},$$

to be taken with the positive sign for the radius of curvature in curves expressed in polar co-ordinates; noticing, that r stands for the *radius vector* in the polar equation.

EXAMPLES.

1. To find the radius of curvature of the spiral of Archimedes, its equation being $r = a\theta$.

Since $\frac{dr}{d\theta} = a$, we shall have $\frac{d^2r}{d\theta^2} = 0$, and thence

$$N = \left\{\left(\frac{dr}{d\theta}\right)^2 + r^2\right\}^{\frac{1}{2}} = a\,(1 + \theta^2)^{\frac{1}{2}}$$

and $$r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\,\frac{d^2r}{d\theta^2} = a^2\,(2 + \theta^2),$$

give $r' = \frac{a\,(1 + \theta^2)^{\frac{3}{2}}}{2 + \theta^2}$ for the radius of curvature.

2. To find the radius of curvature of the spiral whose equation is $r = a\theta^n$.

Since $\frac{dr}{d\theta} = na\theta^{n-1}$, we have

$$N = a\,(\theta^{2n} + n^2\theta^{2n-2})^{\frac{1}{2}} = a\theta^{n-1}\,(n^2 + \theta^2)^{\frac{1}{2}},$$

and $$N^3 = a^3\theta^{3n-3}\,(n^2 + \theta^2)^{\frac{3}{2}};$$

also, since $\frac{d^2r}{d\theta^2} = n(n-1)a\theta^{n-2}$, we shall have

$$r^2 + 2\left(\frac{dr}{d\theta^2}\right) - r\frac{d^2r}{d\theta^2} = a^2\theta^{2n} + 2n^2a^2\theta^{2n-2} - n(n-1)a^2\theta^{2n-2}$$
$$= a^2\theta^{2n-2}(\theta^2 + n^2 + n).$$

Hence we readily get $r' = a\theta^{n-1}(n^2+\theta^2)^{\frac{3}{2}} \div (\theta^2 + n^2 + n)$ for the required radius of curvature.

3. To find the radius of curvature of the logarithmic spiral, its equation being $r = a^\theta$.

Since $\frac{dr}{d\theta} = a^\theta \log a$ and $\frac{d^2r}{d\theta^2} = a^\theta (\log a)^2$, we easily get

$$N^3 = a^{3\theta}[1 + (\log a)^2]^{\frac{3}{2}}$$

$$r^2 + 2\left(\frac{dr}{d\theta}\right)2 - r\frac{d^2r}{d\theta^2} = a^{2\theta}[1 + (\log a)^2];$$

consequently, we shall get $r' = a^\theta\sqrt{[1 + (\log a)^2]}$, the same as the normal: noticing, that log a means the hyperbolic logarithm of a.

4. To find the radius of curvature of the curve whose equation is $r = a \cos\theta$.

Here we have $\frac{dr}{d\theta} = -a\sin\theta$, and $\frac{d^2r}{d\theta^2} = -a\cos\theta$, and, of course, $N = \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2} = a$, and thence the radius of curvature $r' = \frac{a^3}{a^2 + a^2} = \frac{a}{2}$.

Remarks.—By referring to the figure at p. 131, it is manifest that $\frac{r}{SN} = \frac{r}{N}$, and $\frac{rN}{SN} = \frac{dr}{Nd\theta}$ (N being the normal), represent the cosine and sine of the angle NSr, the angle made by the radius of curvature r', and the radius vector r; consequently,

$$\frac{-N^3}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}} \times \frac{r}{N} = \frac{-r\left(\frac{dr}{d\theta}\right)^2 - r^3}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}},$$

and
$$\frac{-N^3}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}} \times \frac{dr}{N d\theta} = \frac{-\left(\frac{dr}{d\theta}\right)^3 - r^2\frac{dr}{d\theta}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}},$$

are the projections of the radius of curvature r' on the radius vector r, and a line perpendicular to r; noticing, that the first of these projections is sometimes called the *co-radius of curvature.* Representing

$$r - \left\{\frac{-r\frac{dr}{d\theta^2} - r^3}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}\right\}$$

and
$$\left\{\left(\frac{dr}{d\theta}\right)^3 + r^2\frac{dr}{d\theta}\right\} \div \left\{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}\right\}$$

by y' and x', we shall have

$$r + \frac{r\left(\frac{dr}{d\theta}\right)^2 + r^3}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}} = y',$$

and
$$\frac{-\left(\frac{dr}{d\theta}\right)^3 - r^2\frac{dr}{d\theta}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}} = x',$$

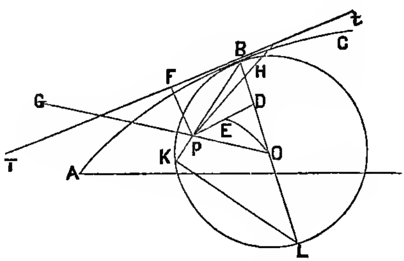
for the rectangular co-ordinates of a point in the evolute of the proposed curve, whose origin is the same as that of the proposed curve.

Thus, in the case of Example 3, we have found in the logarithmic spiral $r' = a^\theta \surd[1 + (\log a)^2] = \text{N}$, the normal, and, of course, we shall have $r - r' \times \frac{r}{\text{N}} = r - r = 0 = y'$; so that the center of the evolute coincides with that of the proposed spiral. And

$$r' \times \frac{dr}{\text{N}d\theta} = \text{N} \times \frac{dr}{\text{N}d\theta} = \frac{dr}{d\theta} = a^\theta \log a = x';$$

or, representing x' by r'', we shall have $r'' = a^\theta \log a$; consequently, since $r = a^\theta$, we shall have $\frac{r''}{r} = \log a$, a constant ratio. Hence, since r'' is perpendicular to r, and has a constant ratio to it, it is manifest that the evolute must be a logarithmic spiral similar to the proposed spiral, their radii vectors making equal angles with their arcs.

(5.) There is another method of finding the radius of curvature, that is often very useful in polar co-ordinates, that may be noticed in this place.



1. Thus, let the curve ABC be supposed to be described by the extremity of PB $= r$ during its angular motion around P in the same plane, in the order of the letters A, B, C; then, if BKL is the circle of curvature at the point B, having O for its center, and its radius OB $=$ R drawn to its point of con-

tact with the curve ABC, or the tangent Tt of the curve at the same point B, by drawing PD and PF perpendicular to BO and the tangent, we shall have the right triangle POD, which gives

$$\mathrm{PO}^2 = \mathrm{PD}^2 + \mathrm{OD}^2 = \mathrm{PD}^2 + (\mathrm{BO} - \mathrm{BD})^2$$

$$= \mathrm{PD}^2 + \mathrm{BO}^2 - 2\mathrm{BO} \times \mathrm{BD} + \mathrm{BD}^2 = \mathrm{PB}^2 - 2\mathrm{BO} \times \mathrm{BD} + \mathrm{BO}^2,$$

which, since PB $= r$ and OB $=$ R, by representing BD $=$ PF by v, becomes $\mathrm{PO}^2 = r^2 - 2v\mathrm{R} + \mathrm{R}^2$; or, denoting PO by r', we shall have $r'^2 = r^2 - 2v\mathrm{R} + \mathrm{R}^2$.

Since the points P and O are fixed for the same circle (by regarding the curve and circle as having a very small common arc), we may take the differential of this equation, on the supposition of the constancy of r' and R, and shall thence get $rdr - \mathrm{R}dv = 0$; which gives $\mathrm{R} = \frac{rdr}{dv}$ for the required expression for the radius of curvature. Admitting the construction of the figure, the equiangular triangles PBD and LBK clearly give the proportion PB : BL : : BD : BK, or its equivalent $r : 2\mathrm{R} :: v : \mathrm{BK} = \frac{2\mathrm{R}v}{r} = \frac{2vdr}{dv} =$ the chord of curvature which passes through the pole, or origin of the co-ordinates; which is a result that is very useful (as is the radius of curvature) in the *doctrine of central forces.* (See Vince's "Fluxions," pp. 149 and 242, together with Newton's "Principia," vol. i., p. 68, &c.)

There are one or two forms of v that are often useful, which it may be well to notice.

2. Thus, if the angle PBF made by r and the tangent Tt, is represented by ϕ; the right triangle PBF gives

$$\mathrm{PF} = v = r \sin \phi.$$

Also, if PG is assumed for the angular axis, and the angle GPB $= \theta$; then we shall, from the principles given at p. 134, get $v = r \times \dfrac{rd\theta}{\sqrt{(r^2d\theta^2 + dr^2)}} = \dfrac{r^2}{\sqrt{\left\{r^2 + \left(\dfrac{dr}{d\theta}\right)^2\right\}}}$.

From this expression, we readily get

$$dv = \frac{r^3dr + 2rdr\left(\dfrac{dr}{d\theta}\right)^2 - r^2dr\,\dfrac{d^2r}{d\theta^2}}{\sqrt{\left\{r^2 + \left(\dfrac{dr}{d\theta}\right)^2\right\}}};$$

consequently, $R = \dfrac{rdr}{dv}$ will be reduced to the form

$$R = \frac{-\left\{r^2 + \left(\dfrac{dr}{d\theta}\right)^2\right\}^{\frac{3}{2}}}{r^2 + 2\left(\dfrac{dr}{d\theta}\right)^2 - r\,\dfrac{d^2r}{d\theta^2}},$$

which agrees with the form of r', the radius of curvature, found at p. 179; noticing, that the radius of curvature must be taken with the positive sign.

Hence, it follows that the equation of the evolute of the proposed curve may be found from the expressions for y' and x', given in the remarks at p. 182. It may be added, that having found R, we can easily find PO or v', from the triangle POB, and also the angle BPO; consequently, the evolute can be constructed by points.

EXAMPLES.

1. To find the radius of curvature in the ellipse when referred to polar co-ordinates, the origin being at the focus.

Taking a and b for the half major and minor axes, we have, from a well-known property of the curve, the equation $\dfrac{2a - r}{r} = \dfrac{b^2}{v^2}$; whose differential gives $\dfrac{adr}{r^2} = \dfrac{b^2dv}{v^3}$, and thence

$\frac{rdr}{dv} = \text{R} = \frac{r^3}{v^3} \times \frac{b^2}{a}$. If we put $p = \frac{b^2}{a}$ = the semiparameter of the major axis, this becomes $\text{R} = \frac{r^3}{v^3} \times p$; which is easily shown to agree with the value of the radius of curvature found at p. 167.

This expression will enable us to find the radius of curvature either of the *ellipse*, *hyperbola*, or *parabola*, by putting p for the *parameter of the major axis*, and observing that r *equals the distance of the point of the curve whose radius of curvature is to be found from the focus* or origin of the co-ordinates, and that v *equals the perpendicular from the focus to the tangent to the curve at the same point.*

2. To find the radius of curvature of the *parabolic spiral* whose equation is $r = a\theta^{\frac{1}{2}}$.

By taking the differentials, we have $\frac{dr}{d\theta} = \frac{1}{2} a\theta^{-\frac{1}{2}}$, and thence get $\left(\frac{dr}{d\theta}\right)^2 + r^2 = \frac{1}{4} a^2\theta^{-1} + a^2\theta = \frac{a^2}{4} \frac{(4\theta^2 + 1)}{\theta}$;

consequently, we shall get

$$\frac{r^2}{\sqrt{\left\{\left(\frac{dr}{d\theta}\right)^2 + r^2\right\}}} = \frac{2a\theta^{\frac{3}{2}}}{(4\theta^2 + 1)^{\frac{1}{2}}},$$

or shall have $v = \frac{2a\theta^{\frac{3}{2}}}{(4\theta^2 + 1)^{\frac{1}{2}}}$. Hence we readily get

$$dv = \frac{a\,(4\theta^2 + 3)\,\theta^{\frac{1}{2}} d\theta}{(4\theta^2 + 1)^{\frac{3}{2}}};$$

consequently, from $rdr = \frac{a^2 d\theta}{2}$ we get $\text{R} = \frac{a\,(4\theta^2 + 1)^{\frac{3}{2}}}{(4\theta^2 + 3)\theta^{\frac{1}{2}}}$, as required.

3. To find the radius of curvature of the equiangular or logarithmic spiral.

From $r \sin \phi = v$, since ϕ is constant or invariable, we get $dv = \sin \phi dr$; and thence from $\frac{rdv}{dv}$, we easily get $\mathrm{R} = \frac{r}{\sin \phi}$, as required. It is hence manifest that R and r are the hypotenuse and leg of a right triangle, having the angle opposite to r equal to ϕ; consequently it follows, as at p. 182, that the evolute must be an equiangular spiral similar to the proposed spiral, and having the same center.

(6). Sometimes, as in finding the radius of curvature in (2), at p. 164, by regarding the evolute as being formed by the intersection of successive normals to the involute, we obtain a convenient method of finding the locus of the intersections of lines or surfaces drawn according to some law, which are sometimes called *consecutive lines and curves.*

EXAMPLES.

1. Suppose we have the equation $xz^2 - yz + a = 0$, such that z is arbitrary; then it is required to find the curve resulting from the elimination of z from the equation, on the supposition of the constancy of y and x when z varies.

By putting the differential coefficient with reference to z equal to naught, we have $2xz - y = 0$, which gives $z = \frac{y}{2x}$. Hence, putting this value of z in the proposed equation, we have $\frac{y^2}{4x} - \frac{2y^2}{4x} + a = 0$, or $y^2 = 4ax$, the equation of a parabola whose parameter equals $4a$.

REMARK.—If the proposed equation had been

$$xz^3 - yz^2 + a = 0,$$

by a similar process we should have found $z = \frac{2y}{3x}$, and thence have obtained $y^3 = \frac{27}{4}ax^2$, the equation of the *semi-cubical parabola*, for the result of the elimination of z from the proposed equation.

2. Given $x'^2 + y'^2 = r'^2$ and $(x - x')^2 + (y - y')^2 = r^2$ for the equations of two circles, to eliminate x' and y' from them.

By taking the differentials of the equations by regarding x' and y' alone as variable, we have $x'dx' + y'dy' = 0$, or $\frac{dy'}{dx'} = -\frac{x'}{y'}$, and from the other equation we in like manner get $\frac{dy'}{dx'} = -\frac{x - x'}{y - y'}$; consequently, from equating these values we get $\frac{x'}{y'} = \frac{x - x'}{y - y'}$, or $x'y = y'x$, which gives $x' = \frac{y'x}{y}$. From the substitution of this value of x' in $x'^2 + y'^2 = r'^2$, we get $y' = \frac{r'y}{\sqrt{(x^2 + y^2)}}$, which reduces $x' = \frac{y'x}{y}$ to $x' = \frac{r'x}{\sqrt{(x^2 + y^2)}}$. Hence, from the substitution of these values of y' and x' in $(x - x')^2 + (y - y')^2 = r^2$, we readily get $x^2 + y^2 - 2r'\sqrt{(x^2 + y^2)} + r'^2 = r^2$, or by extracting the square root of both members of the equation, we have

$$\sqrt{(x^2 + y^2)} - r' = \pm r,$$

or its equivalent,

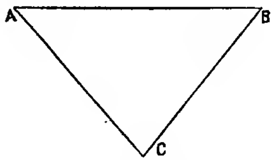
$$x^2 + y^2 = (r' \pm r)^2,$$

equivalent to two circles, represented by $x^2 + y^2 = (r' + r)^2$ and $x^2 + y^2 = (r' - r)^2$. Hence, the series of circles represented by $(x - x')^2 + (y - y')^2 = r^2$, are touched on their outside and inside, or said to be *enveloped* by the circles

$x^2 + y^2 = (r' + r)^2$ and $x^2 + y^2 = (r' - r)^2$, which clearly have the same center as the given circle $x'^2 + y'^2 = r'^2$.

REMARK.—The preceding solution is merely a modification of that given by Young, at pages 146 and 147 of his "Differential Calculus."

3. Supposing ACB to be a triangle, such that the position of AB being changed, the area of the triangle shall be invariable, then it is required to find the curve to which AB shall always be a tangent.



Representing AC and CB by x and y, and assuming $y = ax + b$ for the equation of AB when referred to x and y as axes of co-ordinates, by putting $x = 0$, we shall have $y = b$ or CB $= b$; also, by putting $y = 0$, we have $ax + b = 0$, which gives $x = -\frac{b}{a}$ or AC $= -\frac{b}{a}$.

Because, from the principles of mensuration, the area of the triangle ACB $= \frac{\text{AC} \times \text{CB} \sin \text{ang C}}{2} = -\frac{b^2 \sin C}{2a}$; if we represent this by s, we shall have $s = -\frac{b^2 \sin C}{2a}$ or $a = -\frac{b^2 \sin C}{2s}$. Hence, from the substitution of a, the equation $y = ax + b$, becomes $y = -\frac{b^2 \sin C}{2s} x + b$; whose differential coefficient taken by regarding b alone as variable, gives $-\frac{b \sin Cx}{s} + 1 = 0$, which gives $b = \frac{s}{x \sin C}$. Substituting this value of b in $y = -\dfrac{b^2 \sin C}{2s}x + b$, it becomes $y = -\dfrac{s}{2x \sin C} + \dfrac{s}{x \sin C}$, or $y = \dfrac{s}{2x \sin C}$, or its equivalent $xy = \dfrac{s}{2 \sin C}$, the equation of an hyperbola between its asymptotes, as required, since the curve must clearly always touch the side AB in all its positions.

SECTION VII.

MULTIPLE POINTS, CUSPS OR POINTS OF REGRESSION, ETC.

(1.) *Multiple Points.*—If two or more branches of a curve cross each other at a point, the curve is said to have a multiple point of the first kind; the point being called double, triple, &c., when two, three, &c., branches cross at the point: also, when any number of branches of a curve touch each other at a point, it is said to be a multiple point of the second kind.

If $f(x, y) = 0$ represents the equation of a curve, it is manifest that we may find its multiple points of the first kind, by determining those points of the curve where we have $\left(\frac{dy}{dx}\right)^n = p^n$: such, that n is a positive integer equal to the number of branches that cross each other at the point, and $\frac{dy}{dx} = p$ represents the tangent to any one of the branches at the same point.

It is hence manifest, that to find p, we may differentiate $f(x, y) = 0$ n times successively, by regarding x and y as being independent variables, or by considering dx and dy each as being constant or invariable, when the successive differentials are taken.

EXAMPLES.

1. To find the multiple point of the curve whose equation is $ay^2 + cxy - bx^3 = 0$, at the point whose co-ordinates are

$x = 0$ and $y = 0$, or at the origin of the co-ordinates. By taking the successive differentials, we have

$$2aydy + cxdy + cydx - 3bx^2dx = 0,$$

and $$2ady^2 + 2cdxdy - 6bxdx^2 = 0;$$

which gives

$$\frac{dy^2}{dx^2} + \frac{cdy}{adx} - \frac{3b}{a}x = 0, \quad \text{or} \quad p^2 + \frac{cp}{a} - \frac{3bx}{a} = 0.$$

By putting $x = 0$ in this, we have $p^2 + \dfrac{cp}{a} = 0$, which gives $p = 0$, and $p + \dfrac{c}{a} = 0$, or $p = -\dfrac{c}{a}$; consequently, the curve has a double point at the origin of the co-ordinates, one of whose branches touches the axis of x, since one value of p equals naught, and the other branch makes an angle with the axis of x, whose tangent $= -\dfrac{c}{a}$.

Another Solution.—Solving the equation by quadratics, we have $y = -\dfrac{cx}{2a} \pm \dfrac{cx}{2a}\left(1 + \dfrac{2abx}{c^2} -, \&c.\right)$; whose roots are $y = \dfrac{bx^2}{c} -, \&c.$, and $y = -\dfrac{cx}{a} - \dfrac{bx^2}{c} +, \&c.$

By taking the differentials of these values of y, regarded as a function of x, we get

$$\frac{dy}{dx} = \frac{2bx}{c} -, \&c., \quad \text{and} \quad \frac{dy}{dx} = -\frac{c}{a} - \frac{2bx}{c} +, \&c.;$$

consequently, putting $x = 0$, we get $\dfrac{dy}{dx} = p = 0$, and $\dfrac{dy}{dx} = -\dfrac{c}{a}$, the same as before.

2. To find the multiple points of $y^2 = (x - a)^2x$.

By taking the differentials, we have

$$2ydy = 2(x-a)xdx + (x-a)^2dx,$$

which is satisfied so as to leave dy and dx undetermined, by putting $y=0$ and $x-a=0$ or $x=a$; consequently, if there is a multiple point, it must evidently be at the point represented by $y=0$ and $x=a$.

To find whether $y=0$ and $x=a$ correspond to a multiple point, we differentiate

$$2ydy = 2(x-a)xdx + (x-a)^2dx,$$

by regarding dx and dy as constant, and get

$$2dy^2 = 2xdx^2 + 4(x-a)dx^2,$$

which, by putting $x=a$, reduces to $\left(\frac{dy}{dx}\right)^2 = a$, or $dy = \sqrt{a}$ and $\frac{dy}{dx} = -\sqrt{a}$; consequently, a double point exists at the point whose co-ordinates are $y=0$ and $x=a$.

3. To find the multiple points of the curve whose equation is $y^2 = b^2x + 2bx^2 + x^3$.

Here we have $2ydy = b^2dx + 4bxdx + 3x^2dx$, which is satisfied so as to leave dy and dx undetermined by putting $y=0$ and $b^2+4bx+3x^2=0$, or $x=-b$. Hence, as in the preceding example, we have $2dy^2 = 4bdx^2 + 6xdx^2$, or $\left(\frac{dy}{dx}\right)^2 = 4b+6x$; or, putting $x=-b$, we have $\left(\frac{dy}{dx}\right)^2 = -2b$; consequently, $\frac{dy}{dx} = \sqrt{-2b}$ and $\frac{dy}{dx} = -\sqrt{-2b}$, which is a double point when b is negative. If, however, b is positive, the point represented by $y=0$ and $x=b$, must clearly be detached from all the other points of the curve, though connected with them by the same equation; and such a point is called an *isolated* or *conjugate point.* (See "Cal. Dif.," p. 101, of J. L. Boucharlat; and Young, p. 150.)

4. To find the multiple points of the curve, whose equation is $y^3 = (x-a)^3x^2$.

By taking the differentials, we have

$$3y^2dy = 3\,(x-a)^2x^2dx + 2\,(x-a)^3xdx,$$

which, by putting $y=0$ and $x=a$, leaves dy and dx undetermined; consequently, if there is a multiple point, it must clearly correspond to $y=0$ and $x=a$. Hence, taking the successive differentials, regarding dx and dy as constant, we readily get $\left(\frac{dy}{dx}\right)^3 = x^2$, or, putting a for x, we have

$$\frac{dy}{dx} = p = \sqrt[3]{a^2}.$$

Since this has but one real root, it clearly results that the point corresponding to $y=0$ and $x=a$ is not a multiple point.

Remarks.—It may be shown, in much the same way, that the equation $y^n = (x-a)^nx^m$, when n is an odd integer, can not have a multiple point; and that when n is an even integer, it has a double point.

5. To find the multiple point of $y^2 = (x-a)^4x$.

It is easy to perceive, on account of the inequality of the exponents of y and $x-a$, that the curve represented by the proposed equation, can not have a multiple point of the first kind; consequently, we will proceed to determine whether it has a multiple point of the second kind.

Since by putting $x=a$, the equation is satisfied, and gives $y=0$, by taking its differentials we have

$$2ydy = 4\,(x-a)^3xdx + \,(x-a)^4dx,$$

which is also satisfied by putting $x=a$ and $y=0$, and by taking the differentials of this by supposing y to be a function of x or dx constant, we have

$$2yd^2y + 2dy^2 = 12\,(x-a)^2xdx^2 + 8\,(x-a)^3dx^2,$$

which is satisfied by putting $x = a$, $y = 0$, and $dy = 0$, which leave d^2y undetermined. By taking the differentials as before, we have

$$2yd^3y + 6dyd^2y = 24\,(x-a)\,xdx^3 + 36\,(x-a)^2dx^3,$$

which is also satisfied by putting $x = a$, $y = 0$, $dy = 0$, and leaves d^2y undetermined.

Taking the differentials of this, we have

$$2yd^4y + 8dyd^3y + 6\,(d^2y)^2 = 24xdx^4 + 96\,(x-a)\,dx^4;$$

which, by putting $x = a$, $y = 0$, $dy = 0$, is reduced to

$$6\,(d^2y)^2 = 24adx^4,$$

and is equivalent to $\left(\dfrac{d^2y}{dx^2}\right)^2 = 4a$, or, extracting the square roots of the members of this, we have

$$\frac{d^2y}{dx^2} = 2\sqrt{a} \quad \text{and} \quad \frac{d^2y}{dx^2} = -2\sqrt{a}.$$

It is hence evident that the curve has two branches that touch the axis of x on opposite sides, and each other at the point whose co-ordinates are $x = a$ and $y = 0$, since $dy = 0$ or $\dfrac{dy}{dx} = 0$; and that the order of contact of the branches with the axis of x, and with each other, may be expressed by $\dfrac{d^2y}{dx^2}$.

Otherwise. By taking the square root of the members of the proposed equation, we shall have $y = (x-a)^2\sqrt{x}$. Which, by taking the differentials of its members, gives

$$\frac{dy}{dx} = 2\,(x-a)\sqrt{x} + \frac{1}{2}\,(x-a)^2x^{-\frac{1}{2}},$$

which, by putting $x = a$, gives $\dfrac{dy}{dx} = 0$.

Hence, taking the differentials again, gives

$$\frac{d^2y}{dx^2} = 2\sqrt{x} + 2\,(x-a)\,x^{-\frac{1}{2}} - \frac{1}{4}\,(x-a)^2x^{-\frac{3}{2}};$$

which, by putting $x = a$, reduces to $\frac{d^2y}{dx^2} = \pm\, 2\sqrt{a}$; since the square root ought to be taken with the ambiguous sign $\pm$. Hence, as before, two branches of the curve touch the axis of x on opposite sides, and each other at the point $(y = 0,\ x = a)$ with contact of the order $\frac{d^2y}{dx^2}$, and of course the curve has a double point of the second kind at the point $(y = 0,\ x = a)$.

REMARKS.—It is manifest that this process is more simple than the preceding. And it is manifest that in either method we may take the differentials of the right members of the equations (since it will not affect the results), without taking that of x, the factor of $(x-a)^4$, $(x-a)^3$, &c.

6. To find the multiple point corresponding to $y = 0$ and $x = a$ in the curve whose equation is $y = (x-a)^3\sqrt{x}$.

Since this curve evidently can not have a multiple point of the first kind, we proceed to determine the multiple point of the second kind (by regarding $\sqrt{x}$ as constant), as in the *otherwise* of the preceding example. Hence, we have $dy = 3\,(x-a)^2\sqrt{x}dx$, which for $x = a$ gives $\frac{dy}{dx} = 0$, and shows that the curve touches the axis of x at the point $(y = 0,\ x = a)$. Taking the differentials again, we have $\frac{d^2y}{dx^2} = 6\,(x-a)\sqrt{x}$, which $x = a$ reduces to $\frac{d^2y}{dx^2} = 0$, and, of course, the curve has contact of the order $\frac{d^2y}{dx^2} = 0$ at the point $(y = 0$ and $x = a)$. By taking the differentials again, we have $\frac{d^3y}{dx^3} = 6\sqrt{x}$, which, by putting a for x, and taking $\pm$ before the square root, gives $\frac{d^3y}{dx^3} = \pm 6\sqrt{a}$; consequently, the proposed curve has a double point of the second kind, at the point $(y=0,\ x=a)$, whose order of contact is expressed by $\frac{d^3y}{dx^3}$.

REMARKS.—1. If $y=(x-a)^n x^{\frac{1}{m}} + b$, in which m and n are positive integers, then, it is clear that we may proceed in the same manner as heretofore to find the multiple points.

Thus, if $n = 1$, by taking the differentials we have $\frac{dy}{dx} = x^{\frac{1}{m}}$, or $\left(\frac{dy}{dx}\right)^m = x$; and putting a for x, as heretofore, we have $\left(\frac{dy}{dx}\right)^m = a$, which, when m is an even number, gives $\left(\frac{dy}{dx}\right) = \pm \sqrt[m]{a}$; which clearly gives a double point of the first kind, at the point $(x=a$, and $y=b)$; noticing, if m is an odd integer, that $\frac{dy}{dx} = \sqrt{a}$ is not a multiple, but a single point, since $\sqrt[m]{a}$ can not have but one real odd root, the remaining roots being repetitions, imaginary or impossible.

If n is greater than 1, and m odd, the curve will have a single point, the order of contact being expressed by $\frac{d^ny}{dx^n}$; but if m is even, the curve will have a double point of the second kind, expressed by $\pm\sqrt[m]{a}$, at the point $(x=a$ and $y=b)$; see Young, pp. 151, 152; observing that Mr. Young is clearly incorrect, when he says that a radical of the third degree gives a triple point, and a radical of the mth degree will indicate that m branches of the curve meet at the point ($x = a$ and $y = b$).

2. If $f(x, y) = 0$ represents an explicit function of x and y, then, by finding y in terms of x, after the manner of solving equations, and then proceeding as before, we may find the multiple points, as above.

(2.) *Cusps, or Points of Regression.*—A cusp, or point of regression, is generally considered as a species of double point, at which two touching branches of a curve stop or terminate. If the convexities of the branches touch each other, the point is called *a cusp of the first kind;* while, if the concavity of one branch is touched by the convexity of the other, the point is said to be *a cusp of the second kind.*

It is evident, that the particular co-ordinates of points where cusps exist must be found by particular considerations, and not by the application of Taylor's Theorem; for otherwise the branches would be continued through the cusp, and make it a multiple point, instead of a cusp; against the hypothesis.

REMARK.—When more than two branches of a curve touch each other and stop, it is plain that we may regard them as being cusps, and proceed to treat them in the same way.

EXAMPLES.

1. "To find the cusps of the curve whose equation is

$$y^2 = x^4(1 - x^2)^3 = x^4 - 3x^6 + 3x^8 - x^{10}."$$

It is manifest that the equation is satisfied either by putting $x = 0$ and $y = 0$, or by putting $x = \pm 1$ and $y = 0$. Hence, to determine which of these gives cusps, we may take the differentials of the members of the equation on the supposition of the constancy of dy and dx in the successive differentiations. Hence we shall have

$$ydy = (2x^3 - 9x^5 + 12x^7 - 5x^9)\,dx,$$

which is satisfied by $x = 0$, or $x = \pm 1$ and $y = 0$, while dy and dx remain undetermined. By taking the differentials again, we have

$$dy^2 = (6x^2 - 45x^4 + 84x^6 - 45x^8)\,dx^2,$$

which, by putting $x = 0$ or $x = \pm 1$, gives

$$dy^2 = 0, \quad \text{or} \quad \left(\frac{dy}{dx}\right)^2 = 0;$$

consequently, by extracting the square root, we have

$$\frac{dy}{dx} = 0, \quad \text{and} \quad \frac{dy}{dx} = -0,$$

both when $x = 0$ and when $x = \pm 1$.

It is hence clear that the axis of x is touched on opposite sides at the origin of the co-ordinates, and at the extremities of the axis of x, represented by $x = \pm 1$, or by $x = 1$ and $x = -1$, on the positive and negative sides of the axis. Where it is manifest that the extremities of the axis must be cusps of the first kind, since the convex branches of the curves touch the axis of x and each other at the extremities of the axis, and stop at those points. It is also plain that $x = 0$ and $y = 0$ correspond to a double point of the second kind, since the curve is evidently continued through the origin of the co-ordinates.

Otherwise.—Resuming the equation $y^2 = x^4(1 - x^2)^3$, to find its cusps we may differentiate successively y^2 and $(1 - x^2)^3$ without x^4, or by regarding x as constant, except so far as it is contained in $1 - x^2$. Hence we have

$$2ydy = -6x^5(1 - x^2)^2dx;$$

and differentiating again with reference to $(1-x^2)^2$, as before, we have $2\,(yd^2y+dy^2) = 24x^6\,(1-x^2)\,dx^2$;

and differentiating again with reference to $1-x^2$, we have

$$2\,(yd^3y + 3dyd^2y) = -48x^7\,dx^3.$$

By putting $x = \pm 1$ and $y = 0$ in this equation, we have

$$\frac{dy}{dx} + \frac{d^2y}{dx^2} = \mp 8;$$

consequently, the curvatures at the points $x = \pm 1$ are of higher orders than as found by the preceding method, and they are cusps of the first kind.

To find the multiple point corresponding to x^4, we may reduce $y^2 = x^4\,(1-x^2)$ to $y = x^2\,(1-x^2)^{\frac{3}{2}}$, and take the differentials with reference to x^2 by regarding $(1-x^2)^{\frac{3}{2}}$ as being constant. Hence we shall have

$$dy = 2x\,(1-x^2)^{\frac{3}{2}}\,dx;$$

and in like manner, $d^2y = 2\,(1-x^2)^{\frac{3}{2}}\,dx^2$;

or we shall have $\left(\dfrac{d^2y}{dx^2}\right)^2 = 2^2\,(1-x^2)^3.$

Putting $x = 0$ in this, we have

$$\frac{d^2y}{dx^2} = 2 \quad \text{and} \quad \frac{d^2y}{dx^2} = -2;$$

consequently, the curve has a multiple point of the second kind, at the origin of the co-ordinates; the curvature being clearly of the second degree.

2. To find the cusps of the curve represented by

$$y = x^2 + x^{\frac{3}{2}}.$$

The equation is satisfied by putting $x = 0$ and $y = 0$, or at the origin of the co-ordinates; and by taking the differentials, we have $\dfrac{dy}{dx} = 2x + \dfrac{3}{2}\,x^{\frac{1}{2}}$, which, by putting

$x=0$, gives $\frac{dy}{dx}=0$. It is hence clear that the curve has a cusp of the second kind at the origin of the co-ordinates, its two branches touching the axis of x and each other at the origin.

3. To find the cusps of the curves represented by

$$y^2 = \pm\, x^3.$$

The equations are satisfied by $x=0$ and $y=0$ at the origin; and by taking the differentials, we have

$$2y\,\frac{dy}{dx} = \pm\, 3x^2,$$

which, by taking the differentials again, regarding dy and dx as constant, gives $\quad 2\left(\frac{dy}{dx}\right)^2 = \pm\, 6x;$

which, by putting $x=0$, gives $\frac{dy}{dx} = \pm\, 0.$

It is hence clear that the curves have cusps of the first kind at the origin, the convexities of the curves touching each other.

The cusps may be represented by $0<$ and >0, in which 0 is the origin, the positive values of x being reckoned toward the right; and the first figure corresponds to the sign $+$, while the second corresponds to the sign $-$, in the proposed equations.

4. "To find the cusps of the curve expressed by

$$(y-b)^3 = (x-a)^2."$$

The equation is clearly satisfied by $y=b$ and $x=a$, and by taking the differentials

$$\frac{dy}{dx}=\frac{2}{3}\,\frac{x-a}{(y-b)^2}=\frac{2}{3}\,\frac{1}{(x-a)^{\frac{1}{3}}}$$

and putting $x=a$, we have $\frac{dy}{dx}=$ infinity, or $\frac{dx}{dy}=0$; consequently, setting off $x = a$ from the origin, and drawing the perpendicular $y = b$ to the axis of x through the point thus found, by putting $x = a + h$ we shall have $y = b + h^{\frac{3}{4}}$. By putting positive and negative small values of h in this, we easily get the cusp of the first kind which is formed at the upper extremity of b.

5. "To find the cusps of the curve expressed by

$$y = a + x + (x - b)^{\frac{3}{4}}."$$

The equation is satisfied by putting $x = b$ and $y = a + b$; consequently, putting $x = b + h$, the equation becomes

$$y = a + b + h + h^{\frac{3}{4}}.$$

Taking the differential of this, regarding h as being the independent variable, we have

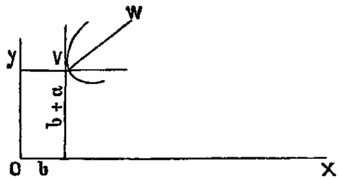
$$\frac{dy}{dx} = 1 + \frac{3}{4} h^{-\frac{1}{4}}, \quad \text{and} \quad \frac{d^2y}{dh^2} = -\frac{3}{16} h^{-\frac{5}{4}}.$$

By putting $h = 0$ in the first of these equations, we have

$$\frac{dy}{dh} = 1 + 0^{-\frac{1}{4}} = 1 + \frac{1}{0^{\frac{1}{4}}} = \text{infinity};$$

consequently, setting off $x = b$ from the origin, and erecting the perpendicular $y = a + b$ to the axis of x, the proposed curve will touch $y = a + b$ at its upper extremity, and it clearly results from $y = a + b + h + h^{\frac{3}{4}}$, that by putting $y = a + b + k$, we shall have $k = h + h^{\frac{3}{4}}$; which clearly shows that h must be positive, since the denominator of the index in $h^{\frac{3}{4}}$ is even, while its numerator is odd, so that k will be imaginary for all negative values of h.

It is hence clear that the proposed curve may be represented by the figure, such that o being the origin of the rectangular axes ox and oy of x and y, b is set from o in the direction of x positive, and the ordinate $b + a$ drawn through

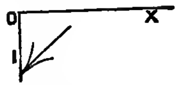


its extremity in the direction of y positive; then, VW being drawn through the upper extremity V of $b+a$ inclined to the axis of x at half a right angle, by drawing parallels to y at the distances h on the axis of x, and setting $+h^{\frac{3}{4}}$ on the line above VW and $-h^{\frac{3}{4}}$ below it, both being set off from VW; it is plain, from what is shown at p. 157, since $\frac{d^2y}{dh^2} = \frac{d^2y}{dx^2}$ is shown, at p. 202, to be of a contrary sign to the ordinate $\pm h^{\frac{3}{4}}$ both above and below VW, that the curve passing through the extremities of the ordinates will give, as in the figure, a curve whose concavity is turned toward its diameter VW; consequently, V can not be a cusp, but it must be what is called *a limit*, since the curve touches the ordinate $b+a$ at V, and lies to the right of the ordinate; since it is plain that h can not be negative, without making the ordinate $\pm h^{\frac{3}{4}}$ imaginary.

6. To find the cusps of the curve whose equation is

$$y = -1 + \frac{x}{3} + x^{\frac{5}{2}}.$$

At the origin, or when $x=0$, we have $x=0$ and $y=-1$, and, by taking the differentials, we have $\frac{dy}{dx} = \frac{1}{3} + \frac{5}{2}x^{\frac{3}{2}}$, which, at the origin, gives $\frac{dy}{dx} = \frac{1}{3}$ for the tangent of the angle which the tangent to the curve at the origin makes

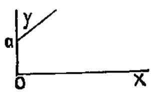


with the axis of x; and since the index in $x^{\frac{5}{2}}$ has an even denominator, x must be positive. Hence it is clear that the curve has a cusp of the first kind below the origin, which may be represented by the annexed figure; o being the origin, and the cusp is at the distance 1, below it.

7. "To find the cusps of the curve whose equation is

$$y = a + x + bx^2 + cx^{\frac{5}{2}}."$$

At the origin, $x = 0$ and $y = a$, also $\frac{dy}{dx} = 1$; consequently, since x must be positive, the curve clearly has a cusp of the second kind above the origin when b is positive. Hence the cusp may be represented by the following figure; in which o is the origin, and the cusp is at the distance a above it. (See Appendix, p. 602, &c.)



SECTION VIII.

PLANE AND CURVE SURFACES.

(1.) *Plane Surfaces.*—1. A plane surface may be represented by referring it to three rectangular axes. Thus, let two equal lines in a plane, called *the axes* of x and y, cut each other perpendicularly at the point O, called *their origin*, then imagine a right line, called *the axis* of z, to be drawn through O at right angles to the plane of x and y, or to that in which they are drawn.

Hence, suppose a plane cuts the axis of z at the distance c, supposed positive, from the origin O, and that it cuts the planes in which the axes of z and x, and those of z and y, lie in two right lines, called *the traces of the plane.* Then from any point in the trace on the plane z and x draw a perpendicular to the plane x, y, and it will cut the axis of x at the distance x, supposed positive, from O, the origin of the co-ordinates; and if a denotes the natural tangent (or the tangent to radius 1) of the angle which the trace makes with the axis of x, we shall clearly have $ax + c$ for the length of the perpendicular to the plane; a being positive when the perpendicular is greater than c, and the reverse. And from the point in the trace, draw a right line parallel to the trace on the plane z, y, and draw a perpendicular, z, from any point in this line to the plane x, y; then, y being the distance of this point from the plane z, x and b the natural tangent of the angle made by the line with the axis of y, we shall, as before, have $z = c + ax + by$ for the equation of the plane; in which b is positive when z is greater than the preceding perpendicular, and the reverse.

2. It is easy to perceive how we may represent the equations of a right line perpendicular to the plane. Thus, imagine planes to be drawn through the perpendicular at right angles to the traces $z = ax + c$, and $z = by + c$; then, from well-known principles of geometry, the common sections of these planes, and the planes z, x and z, y, will be perpendicular to the corresponding traces.

Hence, from what is shown at p. 129, we shall have

$$z = -\frac{x}{a} + c', \quad \text{and} \quad z = -\frac{y}{b} + c'',$$

or $$az + x + c' = 0, \quad \text{and} \quad bz + y + c'' = 0,$$

will represent the equations of the perpendiculars to the traces, or of the perpendicular to the plane.

3. It clearly results from what has been done, that, if we please, we may represent the equation of a plane by the more general form, $$\mathrm{A}x + \mathrm{B}y + \mathrm{C}z + \mathrm{D} = 0;$$

or, if it passes through a point whose co-ordinates are X, Y, and Z, by $\mathrm{A}(\mathrm{X} - x) + \mathrm{B}(\mathrm{Y} - y) + \mathrm{C}(\mathrm{Z} - z) = 0$;

and $$\frac{\mathrm{A}}{\mathrm{C}}(\mathrm{Z}' - z) - (\mathrm{X}' - x) = 0, \quad \frac{\mathrm{B}}{\mathrm{C}}(\mathrm{Z}' - z) - (\mathrm{Y}' - y) = 0,$$

will represent the equations of a perpendicular through the point X', Y', Z', to this plane.

(2.) *Curve Surfaces.*—1. Let $z = f(x, y)$ represent the equation of a curve surface referred to the three rectangular axes of x, y, and z, regarding x and y as being independent variables; then, if $\mathrm{A}x' + \mathrm{B}y' + \mathrm{C}z' + \mathrm{D} = 0$ is the equation of a plane referred to the same axes, which touches the curve surface at the point whose co-ordinates are represented by x', y', and z', it is plain, if the partial differential coefficients

$\frac{dz'}{dx'}$ and $\frac{dz'}{dy'}$ of the surface are represented by p and q, that we must have $p = -\frac{A}{C}$, and $q = -\frac{B}{C}$, the values of the partial differential coefficients $\frac{dz'}{dx'}$ and $\frac{dz'}{dy'}$, given by the tangent plane. Hence, substituting p and q for $-\frac{A}{C}$ and $-\frac{B}{C}$ in the equation of the plane, reduced to the form

$$z' = -\frac{A}{C}x' - \frac{B}{C}y' - \frac{D}{C},$$

we shall have $z' = px' + qy' - \frac{D}{C}$ for the equation of the plane which touches the surface at the point (x', y', z').

Hence we have $Z - z = p(X - x) + q(Y - y)$ for the equation of a plane that passes through the point (X, Y, Z) without the surface, and touches it at any one of its points (x, y, z); and from the equations of the perpendicular,

$$-\frac{A}{C}(Z' - z) + X' - x = 0, \text{ and } -\frac{B}{C}(Z' - z) + Y' - y = 0,$$

to $Z - z = -\frac{A}{C}(X - x) - \frac{B}{C}(Y - y) = p(X - x) + q(Y - y)$, we have $p(Z' - z) + X' - x = 0$, and $q(Z' - z) + Y' - y = 0$, for the equations of a perpendicular through the point (X', Y', Z') to the tangent plane.

If a, b, c denote the angles which the perpendicular, called *the normal*, makes with the axes of x, y, and z, we shall have

$$\cos a = \frac{X' - x}{N}, \ \cos b = \frac{Y' - y}{N}, \text{ and } \cos c = \frac{Z' - z}{N},$$

in which $N = [(X' - x)^2 + (Y' - y)^2 + (Z' = z)^2]^{\frac{1}{2}}$; consequently, substituting $-p(Z' - z)$ and $-q(Z' - z)$

from the equations of the normal, for $X' - x$ and $Y' - y$, we easily get

$$\cos a = \frac{-p}{\sqrt{(p^2+q^2+1)}}, \quad \cos b = \frac{-q}{\sqrt{(p^2+q^2+1)}},$$

and
$$\cos c = \frac{1}{\sqrt{(p^2+q^2+1)}}.$$

It may be added, if z is not explicitly given in terms of x and y, but implicitly by such a function as $u = 0 =$ a function of x, y, and z; then, by taking the differential co-efficients, we shall have

$$\frac{du}{dx} + \frac{du}{dz}\cdot\frac{dz}{dx} = \frac{du}{dx} + \frac{du}{dz}p = 0, \quad \text{and} \quad \frac{du}{dy} + \frac{du}{dz}q = 0,$$

which give $p = -\dfrac{du}{dx} \div \dfrac{du}{dz}$, and $q = -\dfrac{du}{dy} \div \dfrac{du}{dz}$.

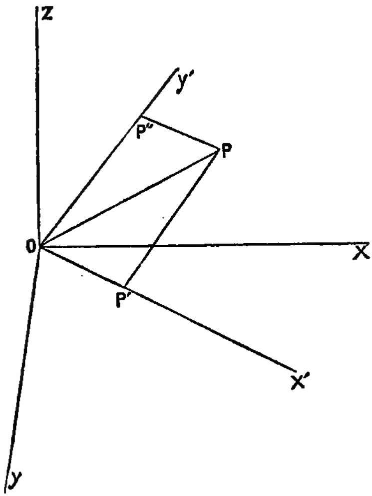
Hence, we must substitute these values of p and q for them, in the preceding equations. (See Young, p. 163.)

2. There are one or two transformations of co-ordinates that are often useful in the determination of the forms of curve surfaces, which we will now proceed to notice.

1st. To change the origin, without altering the directions of the co-ordinates.

Let x, y, and z be the co-ordinates of any point; then, by putting $a + x'$, $b + y'$, and $c + z'$, for x, y, and z, the co-ordinates will be changed into the new co-ordinates x', y', and z', by moving the origin through the distances (here supposed positive) a, b, and c, in directions parallel to the axes of x, y, and z.

2d. To transform the co-ordinates of a point when referred to two rectangular axes, into three co-ordinates referred to three rectangular axes having the same origin.



Let Ox' and Oy' represent the given rectangular axes, and P a point in their plane having OP′ and OP″, denoted by x' and y', for its co-ordinates. Through the origin O of the co-ordinates let Ox be drawn, making the angle xO$x' = \phi$ with Ox', and let the plane of Ox and Ox' make the angle O with the plane of the lines Ox' and Oy'. Then, draw Oy in the plane of Ox' and Ox at right angles to Ox, and Oz at right angles to the plane of Ox and Ox', or Oy. Join OP, then x, y, z, the co-ordinates of P when referred to the rectangular axes of Ox, Oy, and Oz, are clearly the projections of the distance OP on the axes, which are clearly the sums of the projections of OP″ and PP″, or OP″ and OP′, on the same axes. Because Ox' is perpendicular to the lines Oy' and Oz, it is clearly perpendicular to their plane, and in like manner Oy is perpendicular to the plane of the lines Ox and Oz; consequently, the angle made by these planes with each other equals the angle $yOx' = \frac{\pi}{2} - \phi$, by representing the right angle xOy by $\frac{\pi}{2}$, since $\phi =$ the angle xOx'. Since $OP' \cos\phi = x' \cos\phi =$ the projection of x' on the axis of x, and that $OP'' \cos\left(\frac{\pi}{2} - \phi\right) \cos\theta = y' \sin\phi \cos\theta$ equals that of y' on x, we shall have $x = x' \cos\phi + y' \sin\phi \cos\theta$, and in like manner

$$y = x' \cos\left(\frac{\pi}{2} - \phi\right) + y' \cos(\pi - \phi) \cos\theta$$

$$= x' \sin\phi - y' \cos\phi \cos\theta,$$

and $$z = y' \sin\theta.$$

Hence, if we substitute these values of x, y, and z, in the equation of any surface, the resulting equation in x' and y' will give the equation of its section by the plane of x' and y', through the origin of the co-ordinates, and thereby give us a clearer view of the nature of the surface.

Thus, if we take $x^2 + y^2 + z^2 = r^2$, the equation of the surface of a sphere, and make the preceding substitutions, we shall have

$$x'^2 (\cos^2\phi + \sin^2\phi) + y'^2 \cos^2\theta (\cos^2\phi + \sin^2\phi) + y'^2 \sin^2\theta$$
$$= x'^2 + y'^2 = r^2;$$

consequently, the section of a sphere by a plane through its center is a circle whose radius equals the radius of the sphere.

If $x' \cos\phi + y' \sin\phi \cos\theta + a$, $x' \sin\phi - y' \cos\phi \cos\theta + b$, and $y' \sin\theta + c$, are put for x, y, and z, in the equation of a surface, we shall have the equation of a section of the surface, when the origin of the co-ordinates is changed, without altering their directions.

By making these substitutions in the equation of the surface of the sphere, we readily get

$$(x'+a\cos\phi+b\sin\phi)^2+(y'+a\sin\phi\cos\theta-b\cos\phi\cos\theta+c\sin\theta)^2$$
$$=r^2-(a\sin\phi\sin\theta-b\cos\phi\sin\theta-c\cos\theta)^2,$$

for the equation of the section of the surface by the plane of x' and y', which is clearly the equation of a circle whose radius is the square root of the right member of the equation.

If $a=0$, $b=0$, and $\theta=0$, the equation reduces to

$$x'^2+y'^2=r^2-c^2,$$

the cutting plane being at the distance c from the center of the sphere. If $c=r$, the section becomes

$$x'^2+y'^2=0;$$

consequently, we must have $x'=0$ and $y'=0$, and the cutting plane becomes a tangent to the surface of the sphere, and is clearly at right angles to the radius drawn to the point of contact. If c is greater than r, we shall have $x'^2+y'^2=r^2-c^2$, a negative result, which is impossible; consequently, the plane of x' and y' does neither cut nor touch the spheric surface. If $c=0$, the equation becomes $x'^2+y'^2=r^2$, which is called the equation of a great circle of the sphere, while that of $x'^2+y'^2=r^2-c^2$ is called that of a small circle.

(3.) We will now proceed to show how to represent cylindrical, conical, and surfaces of revolution, &c., according to the methods of Monge.

1. Let $x=az+a'$ and $y=bz+b'$ represent the projections of a right line in space, on the planes of the rectangular axes of (x, z) and (y, z), which moves parallel to itself, and during its motion continually passes through the common section of two surfaces, represented by the equations $F(x, y, z)=0$ and $f(x, y, z)=0$; then, the generated surface is said to be of a *cylindrical form*, the moving right

line being called its *generatrix*, while the intersecting surfaces are called its *directrix.*

Because the generatrix is constantly parallel to itself, it is manifest that a and b must be constant or invariable, while a' and b' will vary. But if we eliminate x, y, z from the four equations, which must be constantly coexistent, we shall get an equation that may be expressed by the form $b' = \phi(a')$ = a function of a', or since $a' = x - az$ and $b' = y - bz$, we have $y - bz = \phi(x - az)$ for *the general form of equations of cylindrical surfaces.*

Differentiating the equation separately, by regarding z as a function of x, and then by regarding z as a function of y, we shall have

$$-bp = (1 - ap)\,\phi'(x - az)$$

and $$1 - bq = -aq\phi'(x - az);$$

in which $\phi'(x - az)$ is put for $\frac{d\phi(x - az)}{dz}$, when $\phi(x - az)$ is regarded as being a function of z. Eliminating $\phi'(x - az)$ from the equations, we get $\frac{bp}{1 - bq} = \frac{1 - ap}{aq}$, or its equivalent $ap + bq = 1$, which is called the *general differential equation*, or, more properly, that of partial differential coefficients, of cylindrical surfaces. The same equation results immediately from $Z - z = p(X - x) + q(Y - y)$, the general equation of a tangent plane to cylindrical surfaces. For the equations

$$x = az + a' \quad \text{and} \quad y = bz + b',$$

give $$X - x = a(Z - z) \quad \text{and} \quad Y - y = b(Z - z),$$

which, being substituted in the tangent plaue and the common factor $Z - z$ rejected, becomes $ap + bq = 1$; the same as before. If, as at p. 208, $u = 0$ is an implicit function of x, y, and z, we shall have

$$p = -\frac{du}{dx} \div \frac{du}{dz} \quad \text{and} \quad q = -\frac{du}{dy} \div \frac{du}{dz}$$

which, substituted in the preceding equation, give

$$a\frac{du}{dx} + b\frac{du}{dy} + \frac{du}{dz} = 0;$$

which is, apparently, of a more general form than the preceding equation.

2. To determine the general equation of conical surfaces, we may proceed in much the same way as before, by taking $x - x' = a(z - z')$ and $y - y' = b(z - z')$ for the equations of the right line which constantly passes through its vertex (x', y', z') supposed fixed, and by passing during its motion continually through the directrix $F(x, y, z) = 0$ and $f(x, y, z) = 0$, generates the conical surface.

x', y', and z' being supposed to be known, if we eliminate x, y, and z from the four preceding equations, we shall, as before, get $\frac{y-y'}{z-z'} = \phi\left(\frac{x-x'}{z-z'}\right)$ for the required equation.

Regarding the right member of this equation as being a function of z, and eliminating the function as in the case of cylindrical surfaces, we readily get

$$\frac{(y-y')p}{z-z'-(y-y')q} = \frac{z-z'-(x-x')p}{(x-x')q},$$

or its equivalent $z - z' = p(x - x') + q(y - y')$, which is the equation of the partial differential coefficients of the general equation of conical surfaces; which is clearly the general equation of the tangent plane to the conical surface, as it ought to be. If $u = 0$ is an implicit function of x, y, and z, we shall have

$$p = -\frac{du}{dx} \div \frac{du}{dz} \quad \text{and} \quad q = -\frac{du}{dy} \div \frac{du}{dz},$$

which reduce the preceding equation to

$$(x-x')\frac{du}{dx}+(y-y')\frac{du}{dy}+(z-z')\frac{du}{dz}=0$$

Thus, if for $u=0$ we take $xyz-3y^2x-20=0$, we shall get $yz-3y^2$, $xz-6xy$, and xy, for the values of $\frac{du}{dx}$, $\frac{du}{dy}$, and $\frac{du}{dz}$; or, accenting the letters in these expressions to signify that they correspond to the line of contact of a conical surface with the proposed surface, the above equation, by substitution, becomes

$$(x-x')(y'z'-3y'^2)+(y-y')(x'z'-6x'y')+(z-z')x'y'=0;$$

or, since $x'y'z'-3y'^2x'=20$, it becomes

$$x(y'z'-3y'^2)+y(x'z'-6x'y')+zx'y'=60;$$

which is called the equation of the conical surface, which *envelops the proposed surface*, (x, y, z) being the vertex of the conical surface, or of the *enveloping surface.* The equation being of the second degree in x', y', and z', shows that the line of contact of a conical surface, having its vertex at the point (x, y, z), with the proposed surface, is in a surface of the second degree. It is hence easy to perceive, that if a conical surface envelops a surface of the mth degree, that the line of contact will be in a surface of the $(m-1)$th degree. (See "Application de l'Analyse à la Géométrie," by Monge, pp. 14 and 15; also, see Young, pp. 170 and 171.)

3. To find the general form of the equations of surfaces of revolution.

Let $x=az+a'$, and $y=bz+b'$, represent the equations of the axis of revolution; then, from what is shown at p. 205, $z+ax+by=c$ is the equation of a plane perpendicular to the axis.

Because the perpendicular plane cuts the surface in the circumference of a circle whose center is in the axis, by representing the co-ordinates of the center of a sphere, of which the circle is a section by the perpendicular plane, by a', b', and 0; we shall have $(x - a')^2 + (y - b')^2 + z^2 = r^2$ for the equation of its surface, in which we may suppose a' and b' to be constant, while x, y, z, and the radius r, are variable.

If we substitute the values of $(x - a')^2$ and $(y - b')^2$ from the equations of the axis, in the preceding equation, it becomes $(a^2 + b^2 + 1)\, z^2 = r^2$, in which a and b are constant, or invariable; and substituting ax and by from the equations of the axis in that of the perpendicular plane, it becomes $(a^2 + b^2 + 1)\, z + aa' + bb' = c$.

It is hence manifest, from a comparison of these equations, that we may assume

$$z + ax + by = \phi\, [(x - a')^2 + (y - b')^2 + z^2]$$

for the general equation of surfaces of revolution.

If $a' = 0$ and $b' = 0$, the equation is reduced to

$$z + ax + by = \phi\, (x^2 + y^2 + z^2),$$

which, when $a = 0$ and $b = 0$, or when the axis of revolution coincides with that of z, becomes $z = \phi\, (x^2 + y^2 + z^2)$, or, more simply, we shall have $z = \psi\, (x^2 + y^2)$.

If we regard the right member of the general equation of surfaces of revolution as being a function of z, we shall, from the elimination of the arbitrary function, as heretofore,

get the equation $\dfrac{p + a}{q + b} = \dfrac{x - a' + pz}{y - b' + qz}$,

or its equivalent,

$$(y - b' - bz)\, p - (x - a' - az)\, q + a\, (y - b') - b\, (x - a') = 0;$$

which, when z is the axis of revolution, reduces to $yp - xq = 0$.

The same equations among the preceding partial differential coefficients may readily be obtained from the equations,

$$x - x' + p\,(z - z') = 0, \quad \text{and} \quad y - y' + q\,(z - z') = 0,$$

of the normal to any point (x', y', z') of the surface of revolution, as is evident from the consideration that the normal must pass through the axis of revolution, whose equations must clearly be coexistent with those of the normal.

Hence, eliminating x and y from the equations of the normal by means of the equations of the axis, they will be reduced to

$$az + a' - x' + pz - pz' = (a + p)\,z + a' - x' - pz' = 0,$$

and $$(b + q)\,z + b' - y' - qz' = 0;$$

consequently, eliminating z from these equations, we shall have

$$\frac{a + p}{b + q} = \frac{x' - a' + pz'}{y' - b' + qz'},$$

which agrees with the equation at p. 215, when we use x, y, and z for x', y', and z', as at the place which has been cited; hence, all the preceding results will be obtained, as above.

To illustrate what has been done, let $x = az + a'$, and $y = bz + b'$, represent the equations of a right line revolving around an axis parallel to the axis of z, to find the nature of the surface of revolution described by it.

From $z = \psi\,(x^2 + y^2)$, we clearly get $x^2 + y^2 = \psi' z =$ a function of z; which clearly becomes

$$x^2 + y^2 = (az + a')^2 + (bz + b')^2,$$

from the substitution of the values of x and y from the equations of the revolving line.

To determine the nature of the surface more fully, we shall find the nature of its section by a plane through its axis. Thus, substituting the values of x, y, and z, from p. 210, since $\theta = \frac{\pi}{2} = 90°$, we have $\sin\theta = 1$ and $\cos\theta = 0$; and thence, get $x = x' \cos\phi$, $y = x' \sin\phi$, and $z = y'$. Hence, from the substitution of these values in the preceding equation, since $\sin^2\phi + \cos^2\phi = 1$, we shall have

$$x'^2 = (ay' + a')^2 + (by' + b')^2$$

for the equation of the section of the surface by a plane through the axis of z, which is perpendicular to the plane of x and y. Developing the right member of the equation, we have

$$x'^2 = (a^2 + b^2)\,y'^2 + 2\,(aa' + bb')\,y' + a'^2 + b'^2,$$

or its equivalent,

$$x'^2 = (a^2 + b^2)\left(y' + \frac{aa' + bb'}{a^2 + b^2}\right)^2 - \frac{(aa' + bb')^2}{a^2 + b^2} + a'^2 + b'^2;$$

which, by representing x' and $y' + \frac{aa' + bb'}{a^2 + b^2}$ by X and Y, is readily reducible to

$$X^2 = (a^2 + b^2)\,Y^2 + \frac{(a'b - ab')^2}{a^2 + b^2},$$

the equation of an hyperbola, having $Y = \pm X \div \sqrt{(a^2 + b^2)}$ for the equation of its asymptotes.

Hence it is clear that the equation above found is that of an *hyperboloid of revolution of one sheet*. (See page 20 of Monge's work.)

4. A given curve surface revolves round a given straight line, to find the surface which touches and envelops the moving surface in every position.

The required surface must clearly be a surface of revolution round the given straight line; consequently, the curve of contact of the sought surface and the revolving surface in its first position is evidently a curve whose revolution round the given straight line will generate the required surface. It is hence clear that this question is reducible to that given on page 214, and that p, q, in the revolving curve, must be the same as in the revolving surface in its first position, and that they must satisfy the equation of condition as there found.

Thus, let the revolving surface be that of a spheroid, having $x^2+y^2+n^2z^2=m^2$ for the equation of its surface; and supposing it to revolve round one of its diameters having $x=az$ and $y=bz$ for its equations, when referred to its principal diameters; then from the equation of the surface we shall get $p=-\frac{x}{n^2z}$ and $q=-\frac{y}{n^2z}$.

Because a', b', each equal naught in this example, the equation at page 215 becomes

$$(y-bz)\,p-(x-az)\,q+ay-bx=0;$$

which the substitution of the preceding values of p and q reduce to $ay-bx=0$. Hence the equations of the generating curve of the envelope are expressed by

$$x^2+y^2+n^2z^2=m^2, \quad \text{and} \quad ay=bx;$$

and because the described surface is a surface of revolution, we must also have (see p. 215)

$$ax+by+z=c, \quad \text{and} \quad x^2+y^2+z^2=r^2.$$

From the first and fourth of these equations, we have

$$z^2=\frac{m^2-r^2}{n^2-1};$$

and since the second and third give

$$ay-bx=0 \quad \text{and} \quad ax+by=c-z,$$

we have, by taking the sum of their squares,

$$(a^2 + b^2)\ (x^2 + y^2) = (c - z)^2,$$

which, from $x^2 + y^2 = r^2 - z^2$, is reduced to

$$(a^2 + b^2)\ (r^2 - z^2) = (c - z)^2.$$

Hence, from the substitution of the value of z in this equation, we have

$$(a^2 + b)^2\ (n^2 r^2 - m^2) = [c\sqrt{(n^2 - 1)} - \sqrt{(m^2 - r^2)}]^2,$$

for a conditional equation among the four preceding equations that involve x, y, and z.

Since $r^2 = x^2 + y^2 + z^2$, and $c = ax + by + z$, the preceding equation is equivalent to

$$[n^2\ (x^2 + y^2 + z^2) - m^2]\ .\ (a^2 + b^2) =$$
$$[(z + ax + by)\sqrt{(n^2 - 1)} - \sqrt{(m^2 - x^2 - y^2 - z^2)}]^2,$$

which is the equation of the sought or enveloping surface; agreeing with Mr. Young's result, at p. 176 of his work.

5. When a surface is such that it can be conceived to be spread out on a plane without being torn or rumpled, it is called a *developable surface.*

It is clear that a developable surface may be expressed by means of its tangent plane as follows:

Thus, let $Z - z = p\ (X - x) + q\ (Y - y)$ represent the equation of its tangent plane, which is easily put under the equivalent form $Z = pX + qY + z - px - qy$: in which z, x, y, are the co-ordinates of the point of contact of the plane with the surface; while Z, X, Y, are the co-ordinates of any other point of the plane.

Supposing p and q to be constant, or their total differentials to equal naught, while x, y, and z are changed, since $\frac{dz}{dx} = p$ and $\frac{dz}{dy} = q$, we easily get in differential coefficients, the general equation of developable surfaces. Thus, representing $\frac{dp}{dx}=\frac{d^2z}{dx^2}$, $\frac{dq}{dy}=\frac{d^2z}{dy^2}$, and $\frac{dp}{dy}=\frac{dq}{dx}$, or $\frac{d^2z}{dxdy}=\frac{d^2z}{dydx}$, severally, by r, t, and s, by putting the differentials of p and q equal to naught, we have

$$\frac{dp}{dx}dx+\frac{dp}{dy}dy=0 \quad \text{and} \quad \frac{dq}{dx}dx+\frac{dq}{dy}dy=0,$$

or $$rdx+sdy=0 \quad \text{and} \quad sdx+tdy=0,$$

which give $$\frac{dy}{dx}=-\frac{r}{s} \quad \text{and} \quad \frac{dy}{dx}=-\frac{s}{t}.$$

Equating these values of $\frac{dy}{dx}$, we have

$$\frac{r}{s}=\frac{s}{t}, \text{ or } rt-s^2=0,$$

which is equivalent to

$$\frac{d^2z}{dx^2}\times\frac{d^2z}{dy^2}-\left(\frac{d^2z}{dxdy}\right)^2=0,$$

for the equation of partial differential coefficients of the second order of developable surfaces.

Resuming, $rdx+sdy=0$ and $sdx+tdy=0$, and multiplying the first by dx and the second by dy, by adding the products we have

$$rdx^2+2sdxdy+tdy^2=0,$$

which is called by Monge (at p. 82 of his work), the *characteristic* of developable surfaces.

Because $dz=pdx+qdy$, we shall clearly have

$$d^2z=rdx^2+2sdxdy+tdy^2,$$

consequently, since the right member of the equation equals naught, we shall have $d^2z=0$ in case of a plane; that is, $d^2z=0$ is the *characteristic* of developable surfaces.

Since $d^2z = 0$ belongs to a plane, and that the contact of the tangent plane with the surface is clearly a line, it is evident that all the points of the line may be regarded (see Monge, p. 82) as constituting a plane line.

Because $rt = s^2$, we shall have $s = \sqrt{rt}$, which, being put for s in the characteristic, reduces it to

$$rdx^2 + 2dxdy\sqrt{rt} + tdy^2 = (dx\sqrt{r} + dy\sqrt{t})^2 = 0,$$

whose square root gives

$$dx\sqrt{r} + dy\sqrt{t} = 0, \quad \text{or} \quad \frac{dy}{dx} = -\sqrt{\frac{r}{t}}, \quad \text{or} \quad \frac{dy}{dx} = -\sqrt{\frac{s}{t}};$$

these, when the surface is developable, are

$$\frac{dy}{dx} = -\frac{r}{s} \quad \text{and} \quad \frac{dy}{dx} = -\frac{s}{t}$$

as before shown.

Because $\frac{dy}{dx} = -\frac{r}{s}$, or $-\frac{s}{t}$ = the tangent of the angle which the projection of the line of contact on the plane x, y, makes with the axis of x, it is clear, from what has been done, that the line of contact must be a right line; which may be regarded as being the *generatrix of the developable surface.*

Hence the developable surface (by Monge called the *envelope of the infinitesimal tangent plane*) may be considered as composed of plane elements of unlimited lengths and of infinitesimal breadths, which successively cut each other in right lines. Hence the first of these elementary planes may be turned about its line of common section with the second, until its plane is brought into the same plane with it; and in like manner the plane thus formed, may be turned about the line of common section of the second and third elements, until it is brought into the plane of the third element; and so on to any extent that may be required. It is hence evident that a developable surface may be spread out on a plane without being torn or rumpled.

Because (see pp. 212 and 213) the equations of cylindrical and conical surfaces are represented by

$$ap + bq = 1, \quad \text{and} \quad z' = px' + qy' + z - px - qy,$$

it clearly follows, from what has been done, that they are developable surfaces; since they evidently come under the form $z = pX + qY + z - px - qy$, in which the differentials of p and q are put equal to naught, X, Y, Z are constant, while x, y, z are variable.

REMARKS.—If we assume $z = x\phi(a) + y\psi(a) + a$ to represent the equation of a plane, in which $\phi(a)$ and $\psi(a)$ represent any arbitrary functions of a; then, by putting the first and second differential coefficients taken with reference to a equal to naught, we shall have the forms (see Monge, p. 85)

$$x\phi'(a) + y\psi'(a) + 1 = 0 \quad \text{and} \quad x\phi''(a) + y\psi''(a) = 0.$$

Young (at p. 208 of his "Differential Calculus") says, that for a, in the equation of the plane, the function $f(a)$ ought to be used, since Monge's form excludes those forms comprehended in the form $z = x\phi(a) + y\psi(a) + c$; but this objection is clearly invalid, since, if we please, we may for z put $z - c$, and omit a to suit the case, and we have Mr. Young's form.

It is evident that the equation in x, y, and z, resulting from the elimination of a from

$$z = x\phi(x) + y\psi(a) + a \quad \text{and} \quad x\phi'(a) + y\psi'(a) + 1 = 0,$$

represents a developable surface. For the first equation in virtue of the second, gives

$$\frac{dz}{dx} = p = \phi(a), \quad \text{and} \quad \frac{dz}{dy} = q = \psi(a);$$

consequently, since p and q are functions of a, we shall evidently have $p = \theta(q) =$ a function of q. Hence, we shall

have $$\frac{dp}{dy} = r = \frac{dq}{dx} \times \frac{d\theta(q)}{dq} = s\theta'(q),$$

and $$\frac{dp}{dy} = s = \frac{dq}{dy}\,\theta'(q) = t\theta'(q);$$

consequently, eliminating $\theta'(q)$ from these equations, we get $rt - s^2 = 0$, the equation of developable surfaces, and, of course, the assumed equations jointly represent a developable surface.

Regarding a as being an arbitrary constant, that ought to be retained in the equations, it is clear that the equations may both be regarded as being functions of the characteristic, since the position of the characteristic clearly depends on a.

Hence, the first equation being that of a plane, and the second that of a right line on the plane x, y, it is manifest that the characteristic must be a right line, which is the same as the generatrix of the surface. (See Monge, p. 85.)

If we eliminate a from the equations

$$z = x\phi(a) + y\psi(a) + a,$$

$$x\phi'(a) + y\psi'(a) + 1 = 0,$$

and $$x\phi''(a) + y\psi''(a) = 0,$$

we shall clearly get two equations in terms of x, y, and z, which will clearly be the equations of the line in which the intersections of the successive characteristics must lie, which must evidently be on the developable surface; this line being called by Monge, the *edge of regression of the envelope*, or developable surface. (See Monge, p. 85, and Young, p. 212.)

We will illustrate what has been done by one or two simple examples.

1st. Let $z = xa^2 + ya + b$ represent the variable plane, to find the developable surface and its edge of regression.

Here, by taking the differential coefficients relatively to a, we have the remaining two equations expressed by $2xa + y = 0$, and $2x = 0$. Hence, eliminating a from the first of these and the proposed equation, we get

$$4x(z - b) + y^2 = 0,$$

for the equation of the envelope; which will be found to be a developable surface. If we eliminate a from the three equations, since $2x = 0$, we shall get $z = b$, a point in the axis of z, for the edge of regression.

2d. Let the variable plane be $z = xa^3 + ya^2 + a$; then, the other equations are $3xa^2 + 2ya + 1 = 0$, and $3xa + y = 0$.

Solving the second of these equations by quadratics, we get $a = -\frac{y}{3x} \pm \frac{\sqrt{(y^2 - 3x)}}{3x}$; which, substituted for a in the proposed equation, will give the envelope or developable surface.

Also, eliminating a from the three equations, since the third gives $a = -\frac{y}{3x}$, we have $3x = y^2$ and $yz = -\frac{1}{3}$, the first being that of a parabola on the plane x, y, and the second that of an hyperbola on the plane y, z, for the equations of the edge of regression.

6. By a twisted surface we mean one described by a right line which is continually changing the plane of its motion.

To represent such a surface, we shall suppose $x = az + a'$ and $y = bz + b'$ to be the equations of the generatrix, which we shall suppose to be continually moving along three given curves of double curvature (or which do not lie wholly in the same plane) as directrices; then, each of these curves will be expressed by two equations, when projected on the rectangular planes of x, z, and y, z.

Hence, if we eliminate x, y, and z, from the equations of the generatrix by the equations of any one of the directrices, we shall have an equation involving a, b, a', and b', as unknown quantities; consequently, if we eliminate x, y, and z, in like manner, from the equations of the generatrix by the equations of the remaining directrices, we shall have two more equations, each involving a, b, a', and b', as unknown quantities. Hence, from the solution of the three equations thus found, any one of the quantities a, b, a', and b', as a, may be supposed to be taken for the independent variable, and each of the others to be a function of it. Thus, for a', b, and b', we may put $\psi(a)$, $\phi(a)$, and $\theta(a)$; which reduce the equations of the generatrix to the forms

$$x = az + \psi(a), \quad \text{and} \quad y = z\phi(a) + \theta(a),$$

in which ψ, ϕ, and θ, that precede a, are used to denote any arbitrary functions of it; so that if $\psi(a)$ is assumed to equal naught, $\phi(a) = a^2$, and $\theta(a) = a^3$, our equations will become $x = az$ and $y = a^2z + a^3$, which, by eliminating a from the second by the first, give $yz^3 = x^2z^2 + x^3$, for the equation of a twisted surface.

Generally, if a is found from one of the equations, $x = az + \psi(a)$ and $y = z\phi(a) + \theta(a)$, and its value substituted in the other, z will become a function of x and y, or we shall have the equation of a surface, since z is a function of x and y; consequently, as heretofore, we shall have $dz = pdx + qdy$, in which x and y are the independent variables.

By taking the differentials of $x = az + \psi(a)$ and $y = z\phi(a) + \theta(a)$, supposing a to be constant, we shall have $dx = adz$ and $dy = \phi(a)\,dz$; which show that, in taking the differential of $dz = pdx + qdy$, when a is regarded as constant, on the supposition of the constancy of dx and dy, or that x and y are the independent variables, we must regard dz as also being constant; consequently, we shall have, in this way, $dpdx + dqdy = 0$, in which dp and dq stand for the total differentials of p and q.

Since (see p. 220)

$$dp = \frac{dp}{dx}\,dx + \frac{dp}{dy}\,dy = rdx + sdy$$

and
$$dq = \frac{dq}{dx}\,dx + \frac{dq}{dy}\,dy = sdx + tdy,$$

we shall have the equation

$$dpdx + dqdy = rdx^2 + 2sdxdy + tdy^2 = 0.$$

Because dx and dy are constant, by taking the total differentials of this equation, we shall have

$$d^2pdx + d^2qdy = drdx^2 + 2dsdxdy + dtdy^2 = 0.$$

If we put
$$\frac{dr}{dx} = u, \quad \frac{dr}{dy} = \frac{ds}{dx} = v,$$

$$\frac{ds}{dy} = \frac{dt}{dx} = w, \quad \text{and} \quad \frac{dt}{dy} = u';$$

then, since
$$dr = \frac{dr}{dx}\,dx + \frac{dr}{dy}\,dy,$$

$$ds = \frac{ds}{dx}\,dx + \frac{ds}{dy}\,dy,$$

and
$$dt = \frac{dt}{dx}\,dx + \frac{dt}{dy}\,dy,$$

the preceding equation is reducible to

$$d^2pdx^2 + d^2qdy^2 = drdx^2 + 2dsdxdy + dtdy^2$$
$$= udx^3 + 3vdx^2dy + 3wdxdy^2 + u'dy^3 = 0,$$

or, we shall have

$$u'\left(\frac{dy}{dx}\right)^3 + 3w\left(\frac{dy}{dx}\right)^2 + 3v\,\frac{dy}{dx} + u = 0,$$

an equation of the third order of partial differential coefficients of twisted surfaces.

From $dx = adz$ and $dy = \phi(a)\,adz$, by division, we get $\frac{dy}{dx} = \frac{\phi(a)}{a}$, which must clearly be the same as found from

$$t\left(\frac{dy}{dx}\right)^2 + 2s\,\frac{dy}{dx} + r = 0,$$

or its equivalent $\left(\frac{dy}{dx}\right)^2 + \frac{2s}{t}\,\frac{dy}{dx} = -\frac{r}{t}$,

whose solution gives

$$\frac{dy}{dx} = \frac{-s \pm \sqrt{(s^2 - rt)}}{t} = a';$$

consequently, we shall have $\frac{\phi(a)}{a} = a'$, or $\phi(a) = aa'$, and the remaining equation becomes, by substitution (see Monge, p. 198),

$$u'a'^3 + 3wa'^2 + 3va' + u = 0.$$

If the three directrices are so given as to enable us to find the forms of $\psi(a)$, $\phi(a)$, and $\theta(a)$, then by finding the value of a in one of the equations

$$x = az + \psi(a) \quad \text{and} \quad y = z\phi(a) + \theta(a),$$

and substituting it for a, in the other, as at p. 225, we shall have an equation in x, y, and z, for the equation of the twisted surface, and what is called the *integral* of the equation of partial differential coefficients of the third degree, given on p. 227; if, however, a can not be eliminated from the equations, the equations, in their undetermined form, must be taken for the integrals.

REMARKS.—It is manifest that whatever may be the natures of the directing lines, we may proceed in much the same way, as has been done, to find the equation of the twisted surface described by the motion of the generatrix.

SECTION IX.

CURVATURE OF SURFACES, AND CURVES OF DOUBLE CURVATURE.

(1.) *Curvature of Surfaces.*—Let

$$z' - z = p\,(x' - x) + q\,(y' - y)$$

represent the equation of a tangent plane at a point of a curve surface whose co-ordinates are x, y, and z; then, from what is shown at p. 207,

$$x - \mathrm{X} + p\,(z - \mathrm{Z}) = 0 \quad \text{and} \quad y - \mathrm{Y} + q\,(z - \mathrm{Z}) = 0,$$

are the equations of the normal to the curve surface, at the same point. By taking the differential of the tangent plane, supposing x, y, z, alone to vary, we have $dz = pdx + qdy$; and from

$$dp = \frac{dp}{dx}\,dx + \frac{dp}{dy}\,dy \quad \text{and} \quad dq = \frac{dq}{dx}\,dx + \frac{dq}{dy}\,dy,$$

or (see pp. 220 and 226),

$$dp = rdx + sdy \quad \text{and} \quad dq = sdx + tdy.$$

Taking the differentials of the normals, supposing X, Y, Z, not to vary, we have

$$dx + pdz + (z - \mathrm{Z})\,dp =$$

$$dx + p^2dx + pqdy + (z - \mathrm{Z})\,(rdx + sdy) = 0,$$

and $$dy + pqdx + q^2dy + (z - \mathrm{Z})\,(sdx + tdy) = 0;$$

which are equivalent to

$$1 + p^2 + (z - Z) r + [pq + (z - Z) s] \frac{dy}{dx} = 0,$$

and $$pq + (z - Z) s + [1 + q^2 + (z - Z) t] \frac{dy}{dx} = 0.$$

Eliminating $\frac{dy}{dx}$ from these, we have

$$[1 + p^2 + (z - Z) r] [1 + q^2 + (z - Z) t] \\ = [pq + (z - Z) s] [pq + (z - Z) s],$$

or $$(z - Z)^2 (rt - s^2) + (z - Z) \times \\ [(1 + q^2) r - 2pqs + (1 + p^2) t] + p^2 + q^2 + 1 = 0;$$

and the elimination of $z - Z$ from the same equation, gives

$$\left(\frac{dy}{dx}\right)^2 [(1 + q^2) s - pqt] + \frac{dy}{dx} \\ [(1 + q^2) r - (1 + p^2) t] - (p^2 + 1) s + pqr = 0.$$

These formulas may be much simplified by supposing the tangent plane at the point (x, y, z) to be parallel or coincident with the plane x, y, imagined, to assist the imagination, to be horizontal, the concavity (or hollow) of the surface being turned upward, and the axis of z vertical, its positive value being reckoned upward; then, $p = \frac{dz}{dx}$ and $q = \frac{dz}{dy}$ will evidently be reduced to naught, and the formulas will be reducible to

$$(z - Z)^2 + \frac{r + t}{rt - s^2} (z - Z) + \frac{1}{rt - s^2} = 0,$$

and $$\left(\frac{dy}{dx}\right)^2 + \frac{r - t}{s} \frac{dy}{dx} - 1 = 0.$$

Solving these equations by quadratics, we shall have

$$z - Z = \frac{-(r + t) \pm \sqrt{(r - t)^2 + 4s^2}}{2 (rt - s^2)}$$

and $$\frac{dy}{dx} = \frac{-(r-t) \pm \sqrt{(r-t)^2 + 4s^2}}{2s};$$

which, on account of the ambiguous signs, clearly show that $z - Z$ and $\frac{dy}{dx}$, each admit of two values.

Because $$\frac{dy}{dx} = \frac{-(r-t) + \sqrt{(r-t)^2 + 4s^2}}{2s}$$

and $$\frac{dy'}{dx} = \frac{-(r-t) - \sqrt{(r-t)^2 + 4s^2}}{2s},$$

evidently represent the natural tangents of the angles which two vertical sections of the surface, by planes through the axis of z, make with the plane of the axes of x and z; by taking the product we shall have $\frac{dy}{dx} \times \frac{dy'}{dx} = -1$, consequently, if A stands for the angle whose tangent is $\frac{dy}{dx}$, $A + 90°$ must stand for the angle whose tangent is $\frac{dy'}{dx}$, since $\tan A \times \tan (A + 90°) = \frac{\sin A}{\cos A} \times \frac{\cos A}{-\sin A} = -1$, and, of course, the two planes passing through the axis of z cut each other perpendicularly.

If we represent $z - Z$ by R, we shall have for the equation $$z - Z = \frac{-(r+t) \pm \sqrt{(r-t)^2 + 4s^2}}{2(rt - s^2)},$$

the transformed equation

$$R = \frac{r + t \mp \sqrt{(r-t)^2 + 4s^2}}{2(rt - s^2)};$$

which clearly represent the radii of curvature of the preceding perpendicular sections at their point of contact with the plane of the axes of x and y. Representing these radii separately by R and R′, and taking the sum of their reciprocals, we have

$$\frac{1}{R}+\frac{1}{R'}=\frac{2\,(rt-s^2)}{r+t-\sqrt{(r-t)^2+4s^2}}+$$

$$\frac{2\,(rt-s^2)}{r+t+\sqrt{(r-t)^2+4s^2}}=r+t.$$

Because $r=\frac{dp}{dx}=\frac{d^2z}{dx^2}$ and $t=\frac{dq}{dy}=\frac{d^2z}{dy^2}$,

and that $\frac{dx^2}{d^2z}$ and $\frac{dy^2}{d^2z}$ are clearly the radii of curvature of the vertical sections of the surface which pass through the axes of x and y, it clearly follows that $r+t$ expresses the sum of the reciprocals of these radii. Consequently, since the position of the axes of x and y in the plane of x, y, is arbitrary, it clearly follows that the sum of the reciprocals of the radii of curvature of any two vertical planes through the axis of z, which cut each other perpendicularly, is always equal to $\frac{1}{R}+\frac{1}{R'}$; and of course the sum of the reciprocals of the radii of any two such sections, is always equal to the sum of the reciprocals of the radii of any other two.

If, according to custom, we represent the curvature of the circumference of a circle by the reciprocal of its radius, we shall have *the sum of the curvatures in any two vertical sections that pass through the axis of* z, *and cut each other perpendicularly, equal to the sum of the curvatures in any other two vertical sections that pass through the axis of* z, *and cut each other perpendicularly.* It is hence clear, that if the curvature in one of two perpendicular planes is a maximum, that it must be a minimum in the other plane, and *vice versa.* It is also clear that R and R′—called the *principal radii*—are such, that the first is less than any other radius of curvature, while the other is greater than any other radius.

If we take the principal sections for the axes of co-ordinates, then

$$\frac{dy}{dx} = \frac{-(r-t) + \sqrt{(r-t)^2 + 4s^2}}{2s} = \frac{2s}{r - t + \sqrt{(r-t)^2 + 4s^2}}$$

and

$$\frac{dy'}{dx} = \frac{-(r-t) - \sqrt{(r-t)^2 + 4s^2}}{2s} = \frac{2s}{r - t - \sqrt{(r-t)^2 + 4s^2}},$$

given at p. 230, may be taken for the tangents of the angles which a pair of perpendicular planes through the axis of z makes with the first of the principal planes through the axis of x.

If the perpendicular planes are brought to coincidence with the axes of co-ordinates, we shall have $\frac{dy}{dx} = 0$; and of course, from the first of the preceding formulas, we must have $s = 0$. Hence, putting $s = 0$ in

$$R = \frac{r + t - \sqrt{(r-t)^2 + 4s^2}}{2(rt - s^2)} \text{ and } R' = \frac{r + t + \sqrt{(r-t)^2 + 4s^2}}{2(rt - s^2)},$$

they will become $R = \frac{1}{r}$ and $R' = \frac{1}{t}$ for the radii of curvature of the principal perpendicular sections.

Supposing R″ to stand for the radius of a perpendicular section through the axis of z, which makes an angle whose tangent $= \frac{dy}{dx}$ with the axis of x, then we shall clearly have

$$R'' = \frac{dx^2 + dy^2}{d^2z} = \frac{dx^2 + dy^2}{rdx^2 + tdy^2},$$

since $$s = 0, \text{ or } R'' = \frac{1 + \left(\frac{dy}{dx}\right)^2}{r + t\left(\frac{dy}{dx}\right)^2};$$

which, by supposing $\frac{dy}{dx} = \tan\phi$, becomes

$$R'' = \frac{1 + \tan^2\phi}{r + t\tan^2\phi} = \frac{1}{r\cos^2\phi + t\sin^2\phi};$$

which, by putting for r and t their values $\frac{1}{R}$ and $\frac{1}{R'}$, is easily reduced to

$$R'' = \frac{R'R}{R'\cos^2\phi + R\sin^2\phi},$$

which gives $$\frac{1}{R''} = \frac{\cos^2\phi}{R} + \frac{\sin^2\phi}{R'};$$

which clearly show that if $R = R'$, we shall have $R'' = R$, so that the radii of all the sections through the axis z equal each other.

If R is positive, and R' negative, the preceding value of R'' will be reduced to

$$R'' = \frac{-R'R}{-R'\cos^2\phi + R\sin^2\phi} = \frac{R'R}{R'\cos^2\phi - R\sin^2\phi}$$

and $$\frac{1}{R''} = \frac{\cos^2\phi}{R} - \frac{\sin^2\phi}{R'};$$

noticing, that what is here done corresponds to a circular wheel with a groove in its circumference, R' representing the radius of the wheel whose convexity is turned upward, and R the radius of the groove whose convexity is turned downward, and its concavity upward.

If $$R\sin^2\phi = R'\cos^2\phi \quad \text{or} \quad \tan\phi = \pm\sqrt{\frac{R'}{R}},$$

we shall have $$R'' = \frac{R'R}{0} = \text{infinity};$$

consequently, $\tan\phi = \sqrt{\frac{R'}{R}}$ and $\tan\phi = -\sqrt{\frac{R'}{R}}$ indicate two right lines on the surface drawn to make angles with the axis of x, or width of the groove, passing through its center and making angles with it, whose tangents are $\sqrt{\frac{R'}{R}}$ and $-\sqrt{\frac{R'}{R}}$ on the positive and negative sides of x positive, and on the positive and negative sides of x negative; the surface between these tangents being clearly concave, while the remaining part of it is evidently convex, so that the tangents separate the concavity and convexity of the surface from each other.

Supposing the right line x' is drawn from the origin of the co-ordinates in the plane of x, y, to the surface, so as to make the angle ϕ with the axis of x (see Young, p. 183); then, by assuming $z = \left(\frac{\cos^2\phi}{2R} \pm \frac{\sin^2\phi}{2R'}\right) x'^2$,

since $\cos^2\phi x'^2 = x^2$ and $\sin^2\phi x'^2 = y^2$,

we shall have
$$z = \frac{x^2}{2R} \pm \frac{y^2}{2R'},$$
which is the equation of the surface of a paraboloid of the second order.

From
$$z = \left(\frac{\cos^2\phi}{2R} \pm \frac{\sin^2\phi}{2R'}\right), x'^2,$$
we have
$$\frac{x'^2}{z} = \frac{2RR'}{R'\cos^2\phi \pm R\sin^2\phi} = 2R'',$$
and R'' is the radius of curvature of a vertical section of the paraboloid, at the point whose co-ordinates are x and y, as it clearly ought to be.

Hence we perceive how to measure the curvatures at any proposed point of a surface by those of the paraboloid, and that whether the principal curvatures have the same or contrary directions.

REMARKS.—1. Resuming $R'' = \frac{dx^2 + dy^2}{d^2z}$, from page 233, and representing $\sqrt{dx^2 + dy^2}$ by ds, it will become $R'' = \frac{ds^2}{d^2z}$; which is the radius of curvature in a normal section to the curve surface at its point of contact with the plane of the axes x and y. Suppose now a plane making the angle θ with the normal section, also touches ds at the origin of the co-ordinates; then, it is clear that $\frac{ds^2}{d^2z'}$ will be the radius of curvature in the oblique section with the curve surface, in which d^2z' corresponds to d^2z taken in the normal plane. Because d^2z' and d^2z are clearly the hypotenuse and side of a right triangle having θ for their included angle, we shall have

$$d^2z' \cos\theta = d^2z \quad \text{or} \quad d^2z' = \frac{d^2z}{\cos\theta},$$

which reduces $\frac{ds^2}{d^2z'}$ to $\frac{ds^2}{d^2z} \times \cos\theta = R'' \times \cos\theta$;

consequently, the radius of curvature in the oblique section equals the (orthographic) projection of R'' on the plane of the oblique section, which is called the *Theorem of Meusnier.*

2. Resuming the equations of the normal from p. 229, and putting, with Monge,

$$g = rt - s^2,\ h = (1+q^2)\,r - 2pqs + (1+p^2)\,t, \text{ and } k^2 = p^2 + q^2 + 1$$

in the fourth equation at p. 230, we shall have

$$x - X + p\,(z - Z) = 0, \quad y - Y + q\,(z - Z) = 0,$$

and $$(z-Z)^2+\frac{h}{g}(z-Z)+\frac{k^2}{g}=0,$$

whose solution gives $z-Z=\dfrac{2k^2}{h\pm\sqrt{(h^2-4gk^2)}}$.

Supposing R to be the radius of a sphere that touches the curve surface at the point (x, y, z), having X, Y, Z, for the co-ordinates of its center, we shall have

$$(x-X)^2+(y-Y)^2+(z-Z)=R^2$$

for the equation of the spheric surface, which, by substitution from the equation of the normal, becomes

$$R^2=(p^2+q^2+1)(z-Z)^2=k^2(z-Z)^2;$$

consequently, from the substitution of the preceding value of $z-Z$ in this, we readily get

$$R=\frac{2k^3}{h\pm\sqrt{(h^2-4gk^2)}}$$

for the two radii of curvature at any proposed point of the curve surface.

To illustrate what has been done, we will apply it to find the radii of curvature of the surface, whose equation is

$$z=\frac{xy}{A}.$$

Here we have $\dfrac{dz}{dx}=p=\dfrac{y}{A}$ and $\dfrac{dz}{dy}=q=\dfrac{x}{A}$,

which give $$k^2=\frac{x^2+y^2+A^2}{A^2};$$

and from $$\frac{d^2z}{dx^2}=\frac{dp}{dx}=r=0,$$

since p is not a function of x, and, in the same way,

$$\frac{dq}{dy}=t=0,\quad \text{but}\quad s=\frac{dp}{dy}=\frac{dq}{dx}=\frac{1}{A},$$

which give $$h = -2pqs = -\frac{2xy}{A^3},$$

and they also reduce $g = rt - s^2$ to $-s^2 = -\frac{1}{A^2}$.

Hence the equation $(z - Z)z + \frac{h}{g}(z - Z) + \frac{k^2}{g} = 0$,
is easily reduced to

$$(z - Z)^2 + \frac{2xy}{A}(z - Z) = x^2 + y^2 + A^2,$$

whose solution gives

$$z - Z = \frac{-xy \pm \sqrt{[A^4 + A^2(x^2 + y^2) + x^2y^2]}}{A},$$

or $$Z - z = \frac{xy \pm \sqrt{[A^4 + A^2(x^2 + y^2) + x^2y^2]}}{A}$$

$$= \frac{-[A^3 + A(x^2 + y^2)]}{xy \pm \sqrt{[A^4 + A^2(x^2 + y^2) + x^2y^2]}};$$

which gives

$$R^2 = \frac{(A^2 + x^2 + y^2)^3}{\{xy \pm \sqrt{[A^4 + A^2(x^2 + y^2) + x^2y^2]}\}^2},$$

or $$R = \frac{(A^2 + x^2 + y^2)^{\frac{3}{2}}}{xy \pm \sqrt{[A^4 + A^2(x^2 + y^2) + x^2y^2]}},$$

for the expression of the radii of curvature, at any proposed point of the given surface.

If in the preceding value of R we put $x = 0$ and $y = 0$, we get $R = \pm A$ or $R = A$ and $R = -A$ for the radii of curvature at the origin of the co-ordinates, which is clearly that of the vertex of the given surface; since these radii have contrary signs, it is manifest that the principal curvatures of the surface at its vertex, are turned in opposite directions, and it is manifest that like conclusions are applicable to any other point of the proposed surface, but their magnitudes are not equal, as at the vertex.

It is easy to perceive, that by making analogous substitutions to those made in the equations for the radii vectores in the quadratic equation in $\frac{dy}{dx}$ given at p. 230, we may, after the manner of Monge, at pp. 121, &c., of his "Application de l'Analyse à la Géométrie," proceed to find the integral of the equation, and thence to trace out the lines of curvature on any proposed surface, together with the corresponding radii of curvature. We shall not, however, attempt to do this, but shall satisfy ourselves with the following observations.

Thus, from what is shown at p. 230, it results that there are, at any point of a curve surface, two lines of curvature at right angles to each other, such, that the successive normals to the surface in each intersect each other and form a developable surface; the line in which the successive normals intersect being called the *edge of regression of the developable surface*, while the lines in which the developable surfaces cut the proposed surface are called *lines of curvature.*

(2.) *Curves of Double Curvature.*—In treating of curves of double curvature, it will be sufficient to regard them as consisting of indefinitely small arcs, regarded as straight lines; such that (in general) no more than two successive arcs can lie in the same plane.

Suppose then $x - x' + \mathrm{A}\,(y - y') + \mathrm{B}\,(z - z') = 0$, to represent the plane of any two successive sides of the curve, having x, y, z, for the rectangular co-ordinates of the first extremity of the first of the two successive sides, and x', y', z', for the co-ordinates of any other point of the plane; then dx, dy, and dz, being the differentials of x, y, and z, we shall have $dx + \mathrm{A}dy + \mathrm{B}dz = 0$, when we pass from the co-ordinates of the first extremity of the first (short) side to those of its second extremity, or those of the first extremity of the second side; and in passing from the first to the second extremity of the second (short) side, we in like manner get

$$d^2x + \mathrm{A}d^2y + \mathrm{B}d^2z = 0.$$

From the last two of these equations we get

$$\mathrm{A} = \frac{dzd^2x - dxd^2z}{dyd^2z - dzd^2y}, \quad \text{and} \quad \mathrm{B} = \frac{dxd^2y - dyd^2x}{dyd^2z - dzd^2y};$$

and from the substitution of these values of A and B for them in the equation of the plane, it is readily reduced to the form

$$(x - x')(dyd^2z - dzd^2y) + (y - y')(dzd^2x - dxd^2z) + (z - z')(dxd^2y - dyd^2x) = 0,$$

which is sometimes called the *osculating plane* of the point (x, y, z).

Supposing R to be the radius of a circle passing through the extremities of the same two successive short sides, and that the point (x', y', z') is taken at the center, we shall, from the nature of the circle, have the equation

$$\mathrm{R}^2 = (x - x')^2 + (y - y')^2 + (z - z')^2$$

for the first extremity of the first short side; which, in passing to the second extremity of the first side, gives

$$(x - x')\,dx + (y - y')\,dy + (z - z')\,dz = 0,$$

and this, when we pass to the second extremity of the second side, gives

$$(x - x')\,d^2x + (y - y')\,d^2y + (z - z')\,d^2z + dx^2 + dy^2 + dz^2.$$

If ds represents the length of the first side, since dx, dy, dz, are clearly the projections of ds on the axes of x, y, and z, it is easy to show that we must have

$$dx^2 + dy^2 + dz^2 = ds^2;$$

consequently, the last of the preceding equations becomes

$$(x - x')\,d^2x + (y - y')\,d^2y + (z - z')\,d^2z + ds^2 = 0.$$

By successively eliminating $z - z'$, $y - y'$, and $x - x'$, from the preceding equation, by means of the equation

$$(x - x')\,dx + (y - y')\,dy + (z - z')\,dz = 0,$$

we have the equations

$$(x - x')\,(dxd^2z - dzd^2x) + (y - y')\,(dyd^2z - dzd^2y) = dzds^2,$$

$$(x - x')\,(dxd^2y - dyd^2x) + (z - z')\,(dzd^2y - dyd^2z) = dyds^2,$$

and

$$(y - y')\,(dyd^2x - dxd^2y) + (z - z')\,(dzd^2x - dxd^2z) = dxds^2.$$

Hence, supposing x', y', z', in the osculating plane, to correspond to the center of the circle, by adding its square to the squares of the three preceding equations, because the double products destroy each other, we shall have, since

$$(x - x')^2 + (y - y')^2 + (z - z')^2 = R^2$$

and $$dz^2ds^4 + dy^2ds^4 + dx^2ds^4 = ds^6,$$

$$R^2 = \frac{ds^6}{(dxd^2y - dyd^2x)^2 + (dxd^2z - dzd^2x)^2 + (dyd^2z - dzd^2y)^2},$$

for what is sometimes called the square of the radius of the absolute curvature, corresponding to the point (x, y, z).

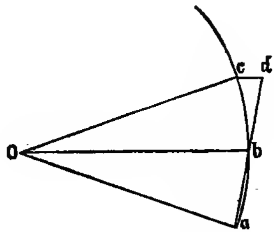
From the development of the squares in the denominator, and omitting the factor $dx^2 + dy^2 + dz^2 = ds^2$, that is common to the numerator and denominator of the resulting fraction,

we have $$R^2 = \frac{ds^4}{d^2x^2 + d^2y^2 + d^2z^2 - d^2s^2}.$$

If ds equals its successive side, ds is constant, and $d^2s = 0$; consequently, we shall have

$$R^2 = \frac{ds^4}{d^2x^2 + d^2y^2 + d^2z^2};$$

which can readily be deduced from the simplest principles of geometry, and that, whether the proposed curve is of single or double curvature.



Thus, let $ab = bc = ds$ represent two successive sides of the polygonal curve, having the circular arc whose center is o, passing through their extremities; then, $oa = ob = oc = \mathrm{R}$ will clearly be the radius of curvature of the proposed curve corresponding to the point a, which we shall suppose to have x, y, and z, for its rectangular co-ordinates. It is clear that the equal straight lines ab and bc subtend the equal arcs ab and bc, which, when ab and bc are indefinitely small, will differ insensibly from them. If ab is produced to d, so as to make $bd = ab$, then, drawing the right line cd, it will evidently be parallel to ob.

It is also manifest that the triangles cbd and obc or oba are equiangular, and give the proportion

$$dc : cb :: cb : bo; \quad \text{which gives} \quad \mathrm{R} = \frac{bc^2}{cd} = \frac{ds^2}{cd}.$$

Also, dx, dy, and dz, the differentials of x, y, and z, the co-ordinates of the point a, are evidently equal to the projections of ab or ds on the axes of x, y, and z, which are clearly equal to the projections of bd on the same axes; and, in like manner, the differentials of the co-ordinates of the point b

equal the projections of bc, or the (algebraic) sum of the projections of bd and dc, on the axes of x, y, and z.

Hence (see p. 2), since the differentials of the co-ordinates of the point b diminished by those of the point a, equal the second differentials of the point a, we get d^2x, d^2y, and d^2z, equal to the sum of the projections of bd and dc diminished by those of ab, on the axes of x, y, and z; consequently, since the projections of bd are destroyed by those of ab, it clearly results that d^2x, d^2y, and d^2z, are the projections of cd on the axes of x, y, and z.

Hence, from the nature of the projection, it being the orthographic (or orthogonal) projection, we shall have

$$cd^2 = d^2x^2 + d^2y^2 + d^2z^2;$$

which reduces $\mathrm{R}^2 = \dfrac{ds^4}{cd^2}$ to $\mathrm{R}^2 = \dfrac{ds^4}{d^2x^2 + d^2y^2 + d^2z^2}$,

which agrees with what is shown at p. 241.

Thus far x, y, and z, have been regarded as being independent of each other; we now propose to consider y and z as being functions of x, expressed by $y = \psi(x)$ and $z = \phi(x)$, the projections of the curve of double curvature on the planes of x, y, and x, z, and shall assume

$$r^2 = (x - x')^2 + (y - y')^2 + (z - z')^2$$

for the radius of a spherical surface, called the *radius of spherical curvature*, supposed to pass through any four successive angles of the polygonal line.

If x, y, z, represent the (rectangular) co-ordinates of the first of the successive angles, by passing to the second angle we get the differential equation

$$(x - x')\,dx + (y - y')\,dy + (z - z')\,dz = 0,$$

or

$$x - x' + (y - y')\frac{dy}{dx} + (z - z')\frac{dz}{dx} = 0;$$

which is clearly the equation of a plane at right angles to the first indefinitely short side that connects the first two of the successive angles, and passes through the center of the spherical surface; or, putting

$$\frac{dy}{dx} = \frac{d\psi(x)}{dx} \quad \text{and} \quad \frac{dz}{dx} = \frac{d\phi(x)}{dx},$$

equal to P and Q, the equation will be more simply expressed by

$$x - x' + (y - y')\,P + (z - z')\,Q = 0.$$

Representing $\frac{dP}{dx}$ and $\frac{dQ}{dx}$ by P′ and Q′, and taking the differential of this equation, we have

$$(y - y')\,P' + (z - z')\,Q' + P^2 + Q^2 + 1 = 0,$$

and from this, by putting $\frac{dP'}{dx}$ and $\frac{dQ'}{dx}$ equal to P″ and Q″, we in like manner get

$$(y - y')\,P'' + (z - z')\,Q'' + 3\,(PP' + QQ') = 0.$$

The first two of these equations, since they represent planes perpendicular to the first and second short sides, and that they intersect in a right line, clearly show the characteristics to be placed on a developable surface, or to have a developable surface for their envelope. And it is evident that the three equations together represent the edge of regression, or the line in which the successive straight lines (that characterize the developable surface) intersect, which is evidently the line in which the centers of spherical curvature lie.

It may be noticed that if the origin of the co-ordinates is taken at a point in the curve having the axis of x for a tangent at the origin, the preceding formulas will be much simplified. For, at the origin we shall have x, y, z, each equal to naught, and P and Q are also reduced to naught, or become infinitesimals. Hence, we shall have

$$x' = 0, \quad y'P' + z'Q' = 1, \quad \text{and} \quad y'P'' + z'Q'' = 0;$$

consequently, we shall have

$$y' = \frac{Q''}{P'Q'' - P''Q'} \quad \text{and} \quad z' = \frac{P''}{P''Q' - P'Q''},$$

and thence $R^2 = y'^2 + z'^2$ reduces to $R^2 = \dfrac{P''^2 + Q''^2}{(P'Q'' - P''C')^2}$.

Again, if in the first two of the formulas we put $\psi(x)$ and $\phi(x)$ for y and z, and eliminate x from the results, we shall obtain an equation in x', y', and z', for the equation of the developable surface noticed above.

If $z - z'$ is eliminated from the same three equations, and $\psi(x)$ is put for y in the two resulting equations, then the elimination of x from these equations gives an equation in terms of y' and x' for the equation of the projection of the edge of regression on the plane of the axes of x and y.

Similarly, by first eliminating $y - y'$ from the equations, putting $\phi(x)$ for z in the results, and eliminating x, we shall get an equation in x' and z' for the equation of the projection of the edge of regression on the plane of the axes of x and z.

If from any point in the common section of the first two perpendicular planes a straight line is drawn in the second plane to the second short side of the curve of double curvature, and produced in the opposite direction to meet the line of common section of the second and third planes, and a line drawn from the point thus found to the third short side of the curve, and produced in the opposite direction to meet, as before, the line of common section of the third and fourth planes, and so on; then, a straight line drawn from the first (assumed) point to the first short side, and continued as a curve in the opposite direction through the points found, will clearly represent the evolute of the proposed curve, regarded as its involute. (See Sec. VI. p. 170.) It is hence easy to perceive that the proposed curve of double curvature has an unlimited number of evolutes.

It is manifest that $\frac{dy'}{dx'} = \frac{y-y'}{x-x'}$ represents a tangent of an evolute of the proposed curve, when projected on the plane of x, y, and that by eliminating $z - z'$ from the equations

$$x - x' + (y - y')\,P + (z - z')\,Q = 0$$

and $(y - y')\,P' + (z - z')\,Q + P^2 + Q^2 + 1 = 0$,

and putting $\psi\,(x)$ for y in the resulting equation, and in

$$\frac{dy'}{dx'} = \frac{y-y'}{x-x'},$$

then, eliminating x from these equations, we shall obtain a differential equation in x' and y', whose integral, according to the principles of the Integral Calculus, will involve one arbitrary constant. The constant will enable us to make the evolute pass through any proposed point; for if the co-ordinates of any point x', y', and z', are represented by a, b, &c., by putting a and b for x' and y' in the integral, we easily get the value of the constant, and thence the integral is determined, so that the projection of the evolute on the plane x, y, passes through that of the proposed point; and in a similar way the projection of the same evolute on the plane x, z, may be determined, and the evolute will be found as required.

For an example we will put

$$y = x \quad \text{and} \quad z = x^2 \quad \text{for} \quad y = \psi\,(x) \quad \text{and} \quad z = \phi\,(x),$$

which give $\frac{dy}{dx} = P = 1, \quad \frac{dz}{dx} = Q = 2x, \quad \frac{dP}{dx} = 0 = P',$

$$\frac{dQ}{dx} = 2 = Q', \quad P'' = 0, \quad \text{and} \quad Q'' = 0.$$

Hence the formulas given at p. 244 become

$$x - x' + y - y' + 2x(z - z') = 0,$$

$$z - z' + 2x^2 + 1 = 0,$$

and

$$Q = 2x = 0,$$

of which the first two give the developable surface, and all three give the edge of regression. If the values of y and z are put for them, in the first two of these equations, they will become

$$2x - x' - y' + 2x^3 - 2xz' = 0 \quad \text{and} \quad 1 - z' + 3x^2 = 0;$$

consequently, putting $x = 0$ in these equations, we have $x' + y' = 0$ and $z' = 1$ for the equations of the edge of regression.

Eliminating x from the first two of the preceding equations, we shall have $\left(\frac{z' - 1}{3}\right)^3 = \left(\frac{x' + y'}{4}\right)^2$ for the equation of the developable surface, which clearly shows that z' must be positive, and not less than 1.

If in $\frac{dy'}{dx'} = \frac{y - y'}{x - x'}$ or $dy'(x - x') = dx'(y - y')$,

we put x for y, we shall have

$$dy'(x - x') = dx'(x - y'),$$

for the equation of the projection of a tangent to an evolute on the plane x, y. And eliminating z' from the equations

$$2x - x' - y' + 2x^3 - 2xz' = 0 \quad \text{and} \quad 1 - z' + 3x^2 = 0,$$

we get

$$x = -\left(\frac{x' + y'}{4}\right)^{\frac{1}{3}};$$

consequently, putting this for x in the preceding differential equation, it becomes

$$dy'\left\{x' + \left(\frac{x'+y'}{4}\right)^{\frac{1}{3}}\right\} = dx'\left\{y' + \left(\frac{x'+y'}{4}\right)^{\frac{1}{3}}\right\}$$

for the differential equation of the projection of an evolute on the plane x, y.

Hence we must find the integral of this equation and determine its constant, so as to suit the nature of the case.

It may be added to what has been done, that if a curve of double curvature has single curvature at one or more of its points, it is said to have lost one of its curvatures, and to have a *single inflection* at such points; while, if it loses both of its curvatures (or becomes rectilineal) at one or more of its points, it is said to have a *double inflection* at such points.

It is manifest from the definition, that the points of single inflection in curves of double curvature, may be found by putting the expression for the radius of spherical curvature equal to ∞, or by putting its reciprocal equal to 0.

Thus, by putting $\frac{P''^2 + Q''^2}{(P'Q'' - P''Q')^2}$ the expression for R^2, given at p. 245, equal to infinity, or putting its reciprocal equal to naught, we have $P'Q'' - P''Q' = 0$, which, from what is done at p. 244, is the same as to put

$$d^2yd^3z - d^3yd^2z = 0,$$

or to find the points of the curve which satisfy the equation

$$\frac{d^3z}{d^2z} = \frac{d^3y}{d^2y}.$$

If this equation can not be satisfied at any point of a curve of double curvature, it clearly can not have a point of single inflection; while, if it can be satisfied at one or more points of the curve, it is manifest that the curve may have single inflections at such points.

To find the points of double inflection, we put the radius of absolute curvature of the curve of double curvature, either equal to 0 or ∞; consequently, from the expression for R^2 given at p. 243, we have

$$d^2x^2 + d^2y^2 + d^2z^2 = 0 \text{ or } \infty.$$

Since this expression has been obtained on the supposition of the constancy of $ds^2 = dx^2 + dy^2 + dz^2$, we put the differential of this equal to naught, which gives

$$dxd^2x + dyd^2y + dzd^2z = 0 \quad \text{or} \quad d^2x = -\frac{dyd^2y + dzd^2z}{dy};$$

which, put for d^2x in the preceding equation, gives

$$d^2y^2 + d^2z^2 + \left(\frac{dyd^2y + dzd^2z}{dx}\right)^2 = 0 \text{ or } \infty.$$

Because this expression consists of the sum of three squares, it is evident that we must satisfy it by putting each of its terms separately, equal to 0 or ∞; consequently, these conditions will be satisfied by

$$d^2y = 0 \text{ or } d^2y = \infty, \quad \text{and} \quad d^2z = 0 \text{ or } d^2z = \infty.$$

These conditions clearly follow from the projections of the curve on the planes x, y, and x, z, which will manifestly be plane curves having (each) the same number of points of inflection; consequently, from the rule given at p. 156, we must have

$$\frac{d^2y}{dx} = 0 \text{ or } \frac{d^2y}{dx} = \infty \text{ or } \frac{dx}{d^2y} = 0 \quad \text{and} \quad \frac{d^2z}{dx} = 0 \text{ or } \frac{dx}{d^2z} = 0,$$

which are clearly equivalent to the preceding conditions.

INTEGRAL CALCULUS.

INTEGRAL CALCULUS.

SECTION I.

(1.) THE INTEGRAL CALCULUS is the reverse of the Differential Calculus; the object being to find the function called the *integral*, from which any proposed differential may be supposed to have been derived.

Thus, since $2xdx$ and $nx^{n-1}dx$ are the differentials of x^2 and x^n, or (more generally) of x^2+c and x^n+c', c and c' being called the *arbitrary constants*, it follows that x^2 and x^n, or, more generally, x^2+c and x^n+c', are the integrals of the proposed differentials.

In like manner, each of the examples given under the rule at p. 5, when an arbitrary constant c (for generality) is added to it, is the integral of its differential; so that the most general integral of $5x^4dx$ is x^5+c, and that of $\frac{na}{b}x^{n-1}dx$ is $\frac{a}{b}x^n+c$ (see examples 1 and 4).

Hence, by reversing the rule at p. 5, it is clear that if we have a differential, such that any power of a variable expression is multiplied by the differential of the expression under the index of the power, which may be multiplied by one or more constants; then, the integral may be found by the following

RULE.

Increase the index of the variable expression by unity, and divide by the increased index and by the differential of expression under the index, and add an arbitrary constant to the result, for the integral of the proposed differential.

Thus, the integral of

$$(a + x^n + y^m)^{p-1} \times \frac{b}{c} p\,(nx^{n-1}dx + my^{m-1}dy)$$

is easily seen to be expressed by

$$\frac{b}{c}(a + x^n + y^m)^p + \text{C},$$

and that of $(\text{F}x)^n d\text{F}x$ is $\dfrac{(\text{F}x)^{n+1}}{n+1} + \text{C}$

(Young's "Integral Calculus," p. 2); and it is manifest that the integrals of all the differentials to which we have referred, can be found by the preceding rule.

(2.) The integral $\dfrac{(\text{F}x)^{n+1}}{n+1} + \text{C}$ of $(\text{F}x)^n d\text{F}x$, admits of a transformation, which we will now proceed to give. Thus, representing the hyperbolic logarithm of Fx by log Fx, we get from the exponential theorem or formula (b), given at p. 51,

$$(\text{F}x)^{n+1} = 1 + (n+1)\log \text{F}x + \frac{(n+1)^2(\log \text{F}x)^2}{1.2} + \frac{(n+1)^3(\log \text{F}x)^3}{1.2.3} +, \&\text{c.};$$

consequently, $\dfrac{(\text{F}x)^{n+1}}{n+1} + \text{C}$ is easily reduced to

$$\frac{1}{n+1} + \log \text{F}x + \frac{(n+1)(\log \text{F}x)^2}{1.2} + \&\text{c.} + \text{C},$$

or, representing $\dfrac{1}{n+1} + \text{C}$ by C′, we shall have

$$\frac{(\mathrm{F}x)^{n+1}}{n+1} + \mathrm{C} =$$

$$\log \mathrm{F}x + \frac{(n+1)(\log \mathrm{F}x)^2}{1.2} + \frac{(n+1)^2(\log \mathrm{F}x)^3}{1.2.3} + \&c. + \mathrm{C}',$$

for the required transformation, a series that evidently converges rapidly, when $n+1$ is small. If $n+1=0$ or $n=-1$, the formula reduces to $\log \mathrm{F}x + \mathrm{C}'$, which, since $n=-1$ is the integral of $\mathrm{F}x^{-1}d\mathrm{F}x = \frac{d\mathrm{F}x}{\mathrm{F}x}$, or using log C for the arbitrary constant, and writing $\int$ before the differential (according to custom) to indicate its integral (the $\int$ being called *the characteristic* of integrals), we shall have

$$\int \frac{d\mathrm{F}x}{\mathrm{F}x} = \log \mathrm{F}x + \log \mathrm{C} = \text{(from the nature of log.)} \log \mathrm{CF}x.$$

Hence, the integral of the differential of a function divided by the function, can be found by the following

RULE.

The integral of the differential of a function divided by the function, equals the hyperbolic logarithm of the function plus an arbitrary constant; or, which comes to the same, the integral equals the hyperbolic logarithm of the product of the function and an arbitrary constant.

REMARK.—Any constant factor (or divisor) of the differential, must be retained in the integral, or the integral must be multiplied (or divided) by it, according to the case.

The rule here given is clearly the reverse of that given at p. 54 (when m the modulus $= 1$), for finding the differential of the hyperbolic logarithm of any expression; deduced from

$$d(\log x) = \frac{dx}{x} \quad \text{or from} \quad d(\log \mathrm{F}x) = \frac{d\mathrm{F}x}{\mathrm{F}x}.$$

Hence, the integrals of the differentials, found under the rule at p. 54, will reproduce the examples, after they are corrected by the introduction of the requisite constants.

Thus, $$\int \frac{dx + dy}{x + y} = \log (x + y) + C,$$

as in example 2, and

$$\int \frac{2mxdx}{a^2 + x^2} = m \int \frac{2xdx}{a^2 + x^2} = m \log (a^2 + x^2) + C;$$

which, if m is the modulus of common logarithms, is equivalent to $\log (a^2 + x^2)$ by using Log before $a^2 + x^2$ to express its common logarithm; as in example 3 (see p. 55).

The integrals of $\frac{dx}{a+x} - \frac{dx}{a-x}$ and $\frac{dx}{x+a} + \frac{dx}{x-a}$ are

$$\log (a + x) + \log (a - x) + C = \log (a^2 - x^2) + C,$$

and $$\log (x + a) + \log (x - a) + \log C = \log C (x^2 - a^2).$$

The integrals of $\frac{bdx}{a + 2x}$ and $\frac{exdx}{a^2 + 3x^2}$, when reduced to proper forms, are

$$\frac{b}{2} \int \frac{2dx}{a + 2x} = \frac{b}{2} \log (a + 2x) + C = \log (a + 2x)^{\frac{b}{2}} + C$$

by the nature of logarithms, and

$$\frac{e}{6} \int \frac{6xdx}{a^2 + 3x^2} = \log C (a^2 + 3x^2)^{\frac{e}{6}}.$$

The integral of

$$\frac{dx}{\sqrt{(x^2 \pm a^2)}} = \left(1 + \frac{x}{\sqrt{(x^2 \pm a^2)}}\right) dx \div [x + \sqrt{(x^2 \pm a^2)}]$$

is expressed by $$\int \frac{dx}{\sqrt{(x^2 \pm a^2)}} = \log C [x + \sqrt{(x^2 \pm a^2)}].$$

(3.) From what is shown in (9), at p. 72, the following table, which is very useful in finding integrals of differentials of certain forms, by means of circular arcs to radius unity, is easily seen to be correct.

$$\int \frac{bdx}{\sqrt{(a^2 - b^2x^2)}} = \sin^{-1}\frac{b}{a}x + C,$$

$$-\int \frac{bdx}{\sqrt{(a^2 - b^2x^2)}} = \cos^{-1}\frac{b}{a}x + C,$$

$$\int \frac{abdx}{a^2 + b^2x^2} = \tan^{-1}\frac{b}{a}x + C,$$

$$-\int \frac{abdx}{a^2 + b^2x^2} = \cot^{-1}\frac{b}{a}x + C,$$

$$\int \frac{adx}{x\sqrt{(b^2x^2 - a^2)}} = \sec^{-1}\frac{b}{a}x + C,$$

$$-\int \frac{adx}{x\sqrt{(b^2x^2 - a^2)}} = \text{cosec}^{-1}\frac{b}{a}x + C,$$

$$\int \frac{bdx}{\sqrt{(a^2x - b^2x^2)}} = \text{versin}^{-1}\frac{2b^2}{a^2}x + C,$$

$$-\int \frac{bdx}{\sqrt{(a^2x - b^2x^2)}} = \text{coversin}^{-1}\frac{2b^2}{a^2}x + C.$$

In using this table, it must be observed, that by the notation $\sin^{-1}\frac{b}{a}x$, is meant an arc of a circle whose radius is unity, and sine $\frac{b}{a}x$, and, in like manner, the remaining expressions are to be understood. (See Young's "Integral Calculus," p. 10, and p. 20 of his "Differential Calculus.")

To perceive the use of the table, take the following

EXAMPLES.

1. To find the integrals of $\frac{dx}{\sqrt{(1-x^2)}}$ and $\frac{2x dx}{\sqrt{(1-x^4)}}$:

Putting $x^2 = y$, the second of these forms becomes $\frac{dy}{\sqrt{(1-y^2)}}$, which is similar to the first form. Hence, since here $a = 1$ and $b = 1$, the integrals will be expressed, according to the first form of the table, by

$$\int \frac{dx}{\sqrt{(1-x^2)}} = \sin^{-1} x + C$$

and $$\int \frac{dy}{\sqrt{(1-y^2)}} = \sin^{-1} y + C = \sin^{-1} x^2 + C.$$

2. To find the integrals of

$$-\frac{dx}{\sqrt{(1-4x^2)}} \quad \text{and} \quad -\frac{dx}{\sqrt{(4-x^2)}}.$$

By putting 1 and 2 for a and b in the first, and the reverse in the second form of the table, we readily get the integrals expressed by

$$-\frac{1}{2}\int \frac{2dx}{\sqrt{(1-4x^2)}} = \frac{1}{2}\cos^{-1} 2x + C$$

and $$-\int \frac{dx}{\sqrt{(4-x^2)}} = \cos^{-1}\frac{x}{2} + C.$$

3. To find the integrals of $\frac{dx}{1+x^2}$ and $\pm\frac{6dx}{4+9x^2}$.

Putting 1 for a and for b in the first of these, and 2 and 3 for them in the second, we get, from the third and fourth forms of the table, the integrals

$$\int \frac{dx}{1+x^2} = \tan^{-1} x + C \quad \text{and} \quad \pm\int \frac{6dx}{4+9x^2} = \begin{matrix}\tan^{-1}\\ \cot^{-1}\end{matrix}\frac{3}{2}x + C.$$

4. To find the integrals of

$$\frac{2dx}{x\sqrt{(9x^2-4)}} \quad \text{and} \quad -\frac{2dx}{x\sqrt{(9x^2-4)}}.$$

By putting 2 and 3 for a and b in the fifth and sixth forms of the table, the integrals will be found to be

$$\int \frac{2dx}{x\sqrt{(9x^2-4)}} = \sec^{-1}\frac{3}{2}x + C$$

and $$-\int \frac{2dx}{\sqrt{(9x^2-4)}} = \text{cosec}^{-1}\frac{3}{2}x + C.$$

5. To find the integrals of

$$\frac{dx}{\sqrt{(x-x^2)}} \quad \text{and} \quad -\frac{dx}{\sqrt{(4x-9x^2)}}.$$

Putting $a=1$ and $b=1$ in the first, and $a=2$ and $b=3$ in the second of these, we readily get

$$\int \frac{dx}{\sqrt{(x-x^2)}} = \text{versin}^{-1}\,2x + C$$

and $$-\frac{1}{3}\int \frac{3dx}{\sqrt{(4x-9x^2)}} = \frac{1}{3}\,\text{coversin}^{-1}\frac{9}{2}x + C.$$

6. To find the integral of

$$\frac{dx}{\sqrt{(ax-bx^2)}}, \quad \text{or of its equivalent} \quad \frac{x^{-\frac{1}{2}}\,dx}{\sqrt{(a-bx)}}.$$

The integral of the first form, from the seventh form of the table, is

$$\int \frac{dx}{\sqrt{(ax-bx^2)}} = \frac{1}{\sqrt{b}}\,\text{versin}^{-1}\frac{2b}{a}x + C,$$

and the integral of the second form, from the first form of the table, is

$$\int \frac{x^{-\frac{1}{2}}\,dx}{\sqrt{(a-bx)}} = \frac{2}{\sqrt{b}}\sin^{-1}\left(\frac{b}{a}\right)^{\frac{1}{2}} x^{\frac{1}{2}};$$

and it is easy to perceive that these integrals are equivalent.

(4.) If a differential consists of two or more terms, such that it has an algebraic sum of factors so related that the differential of each factor is multiplied by the product of all the remaining factors, then, from reversing the rules at pp. 9 and 10, it follows that the product of all the (different) factors, plus an arbitrary constant, will be the integral of the differential.

This rule is easily perceived to be correct by reversing the methods of finding the differentials of the examples at pp. 9 to 11. Thus, from the differentials in example 1, at p. 9, we have

$$\int (xdy + ydx) = xy + C$$

and $\int (3x^2dy + 6yxdx) = 3 \int (x^2dy + ydx^2) = 3x^2y + C.$

And from the fifth and sixth examples, we have

$$a\int (3xy^2z^2dz^2 + 2xz^3ydy + y^2z^3dx) =$$

$$a\int (xy^2dz^3 + xz^3dy^2 + y^2z^3dx) = axy^2z^3 + C,$$

and $$\int (2xy^{-3}dx - 3x^2y^{-4}dy) =$$

$$\int (y^{-3}dx^2 + x^2dy^{-3}) = x^2y^{-3} + C = \frac{x^2}{y^3} + C.$$

Also, from example 4, at p. 10, we have

$$\int\left(\sqrt{(a^2+x^2)} \times - \frac{xdx}{\sqrt{(a^2-x^2)}} + \sqrt{(a^2 - x^2)} \times \frac{xdx}{\sqrt{(a^2+x^2)}}\right)$$

$$= \int [\sqrt{(a^2 + x^2)} \times d\sqrt{(a^2 - x^2)} + \sqrt{(a^2-x^2)}\, d\sqrt{(a^2 + x^2)}]$$

$$= \sqrt{(a^2 + x^2)}\sqrt{(a^2 - x^2)} + C = \sqrt{(a^4 - x^4)} + C;$$

and in the same way, the integral of

$$\frac{dx}{\sqrt{(a^2 + x^2)}} - \frac{x^2dx}{\sqrt{(a^2 + x^2)^3}}$$

is easily found to be $\frac{x}{\sqrt{(a^2 + x^2)}} + C$ as at p. 10.

REMARKS.—1. From the first of these examples, we have

$$\int (xdy + ydx) = \int xdy + \int ydx = xy + C,$$

which gives $\int ydx = xy - \int xdy + C,$

and evidently reduces the integration of ydx to that of xdy; and, in like manner, the integral of xdy is reducible to that of ydx. This process, which is often very useful in finding and simplifying integrals, is called *integration by parts.*

2. To illustrate this method, we will apply it to find the integral of Xdx, supposing X to be a function of x.

Because $Xdx = Xdx + xdX - xdX,$

by taking the integrals of these equals, we have

$$\int Xdx = Xx - \int xdX;$$

and since $dX = \frac{dX}{dx} dx,$

we have $xdX = \frac{dX}{dx} xdx,$

which gives $\int xdX = \frac{dX}{dx} \frac{x^2}{2} - \int \frac{d^2X}{dx^2} \frac{x^2dx}{2}.$

And in like manner, from $\int \frac{d^2X}{dx^2} \frac{x^2dx}{2}$

we have $\frac{d^2X}{dx^2} \frac{x^3}{2.3} - \int d^3X \frac{x^3dx}{2.3},$

and so on, to any extent required.

Hence, from the substitution of these values in

$$\int Xdx = Xx - \int \frac{dX}{dx} xdx,$$

we get

$$\int Xdx = Xx - \frac{dX}{dx} \frac{x^2}{1.2} + \frac{d^2X}{dx^2} \frac{x^3}{1.2.3} - \frac{d^3X}{dx^3} \frac{x^4}{1.2.3.4} + \&c. + C,$$

which is called *the Formula, or Series, of John Bernouilli;* which clearly shows that the proposed integral can always be found, at least in a series.

If we put x, x^2, x^3, &c., successively for X, in the formula, we shall get

$$\int x dx = x^2 - \frac{x^2}{1.2} + C = \frac{x^2}{2} + C,$$

$$\int x^2 dx = x^3 - x^3 + \frac{x^3}{3} + C = \frac{x^3}{3} + C,$$

$$\int x^3 dx = x^4 - \frac{3x^4}{1.2} + x^4 - \frac{x^4}{4} + C = \frac{x^4}{4} + C,$$

and so on; results which are evidently correct.

It may be added, that Maclaurin's Theorem is applicable to the expansion of $\int X dx$. Thus, putting $\int X dx$ for X in Maclaurin's Theorem (*b*), given at p. 17, we shall have

$$\int X dx = (\int X dx) + (X)x + \left(\frac{dX}{dx}\right)\frac{x^2}{1.2} + \left(\frac{d^2X}{dx^2}\right)\frac{x^3}{1.2.3} +, \&c.,$$

for the required expansion; in which, for x, we must put naught in the expressions within the parentheses, and the term $(\int X dx)$ must clearly represent the arbitrary constant. Thus, if we put x^3 for X, the formula becomes

$$\int x^3 dx = C + \frac{x^4}{4};$$

since (x^3), $(3x^2)$, and $(3x2x)$, are reduced to naught, when naught is put for x in them, while the term

$$\left(\frac{d^3X}{dx^3}\right)\frac{x^4}{1.2.3.4} \quad \text{becomes} \quad 3 \times 2 \times 1 \times \frac{x^4}{1.2.3.4} = \frac{x^4}{4}.$$

Mr. Young (at p. 81 of his "Integral Calculus") says, that Maclaurin's Theorem fails to be applicable, when $x = 0$ reduces the preceding coefficients to naught; which is certainly incorrect, since the Theorem is always applicable when it has no infinite term (see p. 17). It may be added, that Maclaurin's Theorem is (generally) more useful in practice than that of Bernouilli.

(5.) By reversing the rule at p. 56, we easily find the integral of an exponential differential, when the exponent of the exponential is alone variable, by the following

RULE.

Divide the differential by the hyperbolic logarithm of the base of the exponential and by the differential of its exponent, and add an arbitrary constant to the result for the integral.

The truth of the rule is manifest by reversing the method of finding the differentials at p. 56.

Thus, from the first example we have

$$\int 2^x \log 2\, dx = 2^x \log 2\, dx \div \log 2\, dx + C = 2^x + C,$$

and

$$\int 3^y \log 3\, dy = 3^y + C.$$

Also $\int a^x \log a dx = a^x + C$ and $\int e^x dx = e^x + C$,

supposing e to be the hyperbolic base.

And from reversing the rule at p. 59, it results that the integral of a cosine of a variable multiplied by the differential of the variable, equals the sine of the variable plus a constant. Also, the integral of minus the sine of a variable multiplied by the differential of the variable, equals the cosine plus a constant.

Thus,
$$\int \cos 2x \times 2dx = \sin 2x + C,$$

and
$$-\int \sin 3x \times 3dx = \cos 3x + C.$$

And $\int \cos 4x dx = \frac{1}{4}\int 4dx \times \cos 4x = \frac{\sin 4x}{4} + C$, and

$$-\int \sin 5x \times 2dx = -\frac{2}{5}\int \sin 5x \times 5dx = \frac{2}{5}\cos 5x + C,$$

and so on, as in reversing the examples at p. 60.

REMARKS.—We might, in like manner, proceed to reverse the rules at p. 60, &c., but we do not think it necessary to consider them any further in this place.

(6.) If any number of differentials are connected together by the signs + and —, it is manifest that they are to be considered as constituting one differential, whose integral requires only one arbitrary constant.

Thus, if we have

$$2axdx + 3bx^2dx - 4cx^3dx,$$

it is to be considered as a single differential, having

$$\int (a2xdx + b \times 3x^2dx - c \times 4x^3dx) = ax^2 + bx^3 - cx^4 + C$$

for its integral, C being the arbitrary constant. Reciprocally, for any expression like

$$\int (ax^3dx + b^5 \times dx - px^6dx +, \&c.),$$

we may, if we please, write

$$a\int x^3dx + b\int x^6dx - p\int x^6dx$$

or $$\int (ax^3dx + bx^5dx) - \int px^6dx, \&c.$$

(7.) It may be added, that the arbitrary constants in integrals, are (generally) to be determined so as to satisfy certain conditions which the integrals must answer.

Thus, if the integral $\int (3x^2dx + 5x^4dx)$ must equal naught when $x = a$, we proceed as follows. By integration we have

$$\int (3x^2dx + 5x^4dx) = x^3 + x^5 + C;$$

consequently, putting a for x in this, we must, from the conditions of the question, have $a^3 + a^5 + C = 0$, which gives $C = -(a^3 + a^5)$ for the value of the constant. Hence the integral, duly corrected, becomes

$$\int (3x^2dx + 5x^4dx) = x^3 + x^5 - (a^3 + a^5);$$

and it is manifest that we must proceed in like manner in all analogous cases.

To signify that an integral, as

$$\int (axdx + bx^2dx - cx^3dx)$$

is to be taken from $x = A$ to $x = B$, we write

$$\int_A^B (axdx + bx^2dx - cx^3dx) = \frac{ax^2}{2} + \frac{bx^3}{3} - \frac{cx^4}{4} + C,$$

which, by putting A for x, gives

$$C = -\left(\frac{aA^2}{2} + \frac{bA^3}{3} - \frac{cA^4}{4}\right),$$

and thence the integral becomes

$$\int_A^B (axdx + bx^2dx - cx^3dx)$$

$$= \frac{ax^2}{2} + \frac{bx^3}{3} - \frac{cx^4}{4} - \left(\frac{aA^2}{2} + \frac{bA^3}{3} - \frac{cA^4}{4}\right),$$

which, by putting B for x in its right member, becomes

$$\int_A^B (axdx + bx^2dx - cx^3dx)$$

$$= \frac{aB^2}{2} + \frac{bB^3}{3} - \frac{cB^4}{4} - \left(\frac{aA^2}{2} + \frac{bA^3}{3} - \frac{cA^4}{4}\right)$$

$$= \frac{a}{2}(B^2 - A^2) + \frac{b}{3}(B^3 - A^3) - \frac{c}{4}(B^4 - A^3),$$

which is called *a definite integral*, because x, in its right member, is determined; consequently, when an integral is taken from one value of its variable to another value of its variable, the integral is *definite* or *determined*, otherwise the integral is *indefinite* or *not fixed.*

(8.) When the integral of a proposed differential is found, it is said *to be integrated;* and when the integral is taken from one value of the variable (x) to any other proposed value, it is said to be integrated from the first to the second value of the variable.

(9.) To aid in what is to follow, and to show the *natures of differentials and integrals* more fully, we will now proceed to give the solution of the following important

PROBLEM.

If x and $y = f(x) =$ a function of x, represent the abscissa and corresponding rectangular ordinate of a plane curve, it is proposed to show how to find the area bounded by the ordinate drawn through the origin of the co-ordinates, by any other ordinate, and the intercepted parts of the axis of x and the curve: supposing the ordinate to be constantly positive between the preceding limits.

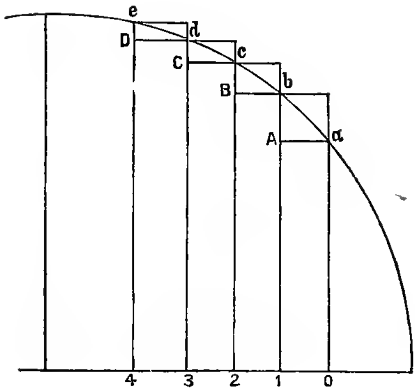
It is clear that we may suppose $f(x)$ to be expressed by

$$A + Bx^a + Cx^b + Dx^c +, \&c.,$$

in which A, B, C, &c., a, b, c, &c., are independent of x, which, for simplicity, we shall suppose to be positive, and that x^a, x^b, x^c, &c., are arranged according to the ascending powers of x.

Let then, in the figure, 0 be the origin of the co-ordinates, and suppose 04 represents any abscissa, and $4e$ the corresponding ordinate; we propose to find the area or quadrature of the curve, bounded by the ordinates $0a$ and $4e = y$, the abscissa $04 = x$, and the portion of the curve ae.

Suppose x to be divided into any number (n) of equal



parts at the points 1, 2, 3, &c., and let x' represent any one of these parts; then, the ordinates corresponding to the points 0, 1, 2, 3, &c., may evidently be expressed by

$$A = y', \quad A + Bx'^a + Cx'^b + \&c. = y'',$$
$$A + B(2x')^a + C(2x')^b + \&c. = y''',$$

and so on to

$$A + B(nx')^a + C(nx')^b + \&c. = y^{n+1}.$$

Allowing the rectangles to be drawn as in the figure, it is easy to perceive that the sum of all the inscribed rectangles will be expressed by

$$(y' + y'' + \ldots . + y^n)\, x' =$$
$$Anx' + B\,[1 + 2^a + 3^a + \ldots . + (n-1)^a]\, x'^{a+1}$$
$$+ C\,[1 + 2^b + 3^b + \ldots . + (n-1)^b]\, x'^{b+1} +, \&c.,$$

as is manifest from the principles of mensuration, while the sum of all the corresponding circumscribed rectangles will be expressed by

$$(y'' + y''' + \dots + y^{n+1})\,x' =$$
$$An x' + B\,[1 + 2^a + 3^a + \dots + n^a]\,x'^{a+1}$$
$$+ C\,[1 + 2^b + 3^b + \dots + n^b]\,x'^{b+1} +, \text{\&c.}$$

It is easy to perceive that the difference between the preceding sums of the circumscribed and inscribed rectangles is expressed by

$$(y^{n+1} - y')\,x' = Bn^a x'^{a+1} + Cn^b x'^{b+1} + \text{\&c.} = (Bx^a + Cx^b + \text{\&c.})\,x',$$

since $nx' = x$. If x' is unlimitedly small, the difference is evidently also unlimitedly small; consequently, since the difference is clearly greater than the difference between the sought area of the curve and the sum of all the inscribed or circumscribed rectangles, it is manifest that, by taking x' sufficiently small, the sum of all the inscribed or circumscribed rectangles may be made to differ from the sought area of the curve by a difference which shall be unlimitedly small. (See Lemma II., Book I., of Newton's "Principia.")

It is clear that what has been done holds good, whether *ae* is a curve or straight line, or even if it is a curve whose convexity is turned toward the line of the abscissæ or the axis of x.

We now propose to put the above expressions for the sums of the inscribed and circumscribed rectangles under more useful forms.

By putting $n - 1 = n'$, it is clear that we may assume

$$1 + 2^a + 3^a + \dots + n'^a = Pn'^{a+1} + Qn'^a + Rn'^{a-1} +, \text{\&c.},$$

and suppose P, Q, &c., to be independent of n' and

$$1 + 2^b + 3^b +, \text{\&c.},$$

clearly admit of like representations. By changing n' into $n' + 1$, and subtracting the assumed equations from the results, we get the identical equations

$$(n'+1)^a = n'^a + an'^{a-1} + \&c. =$$

$$P\left[(a+1)\,n'^a + \frac{(a+1)\,a}{1\,.\,2}\,n'^{a-1} + \&c.\right] +$$

$$a\left[an'^{a-1} + \frac{a\,(a-1)}{1\,.\,2}\,n'^{a-2} + \&c.\right] +, \&c.,$$

and so on; of course, by equating the coefficients of like powers of n', in the members of the equations, we readily get $P = \frac{1}{a+1}$, $Q = \frac{1}{2}$, $R = \frac{a}{3.4}$, $S = 0$, $T = -\frac{a\,(a-1)\,(a-2)}{1.2.3.4.5.6}$, &c.; and so on, for the other representations.

From the substitution of the preceding values of P, Q, &c., by putting for n' its value $n-1$, and expanding the powers of $n-1$ according to the descending powers of n (as heretofore) by the binomial theorem, we get

$$1 + 2^a + 3^a + \ldots + (n-1)^a = \frac{(n-1)^{a+1}}{a+1} + \frac{(n-1)^a}{1\,.\,2} + \&c.$$

$$= \frac{n^{a+1}}{a+1} - \frac{n^a}{1.2} + \frac{an^{a-1}}{3.4} - \frac{a\,(a-1)\,(a-2)\,n^{a-1}}{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6} +, \&c.;$$

and by changing a into b, c, &c., we get the corresponding representations of $1 + 2^b + 3^b + \ldots + (n-1)^b$, and so on.

It may be proper to notice here, that the numbers $Q = -\frac{1}{1.2}$, $R = \frac{1}{3.4}$, and so on, called the numbers of (James) Bernouilli, may easily be calculated to any extent, by solving the equations

$$Q + \frac{1}{1.2} = 0, \quad R + \frac{Q}{1.2} + \frac{1}{1.2.3} = 0,$$

$$S + \frac{R}{1.2} + \frac{Q}{1.2.3} + \frac{1}{1.2.3.4} = 0,$$

and so on. (See p. 98, Vol. III., of Lacroix's "Traité du Calcul Différentiel," etc.)

Hence, from the substitution of the preceding values in the expression for the inscribed rectangles, at p. 267, we shall have

$$(y' + y'' + \dots + y^n)\,x' =$$

$$\left(\text{A}nx' + \frac{\text{B}\,(nx')^a nx'}{a+1} + \frac{\text{C}\,(nx')^b nx'}{b+1} + \text{\&c.}\right) +$$

$$\frac{\text{A}x'}{1.2} - \frac{1}{1.2}\left[\text{A} + \text{B}\,(nx')^a + \text{C}\,(nx')^b + \text{\&c.}\right] x' + \frac{1}{3.4}$$

$$\left[a\text{B}\,(nx')^{a-1} + b\text{C}\,(nx')^{b-1} + \text{\&c.}\right] x'^2 - \frac{1}{1.2.3.4.5.6}$$

$$\left[a\,(a-1)\,(a-2)\,\text{B}\,(nx')^{a-3} + b\,(b-1)\,(b-2)\,\text{C}\,(nx')^{b-3} + \text{\&c.}\right]$$

$$x'^4 + \text{\&c.} = (\text{since } nx' = x)\ \text{A}x + \frac{\text{B}x^{a+1}}{a+1} + \frac{\text{C}x^{b+1}}{b+1} + \text{\&c.}$$

$$+ \frac{\text{A}x'}{2} - \frac{1}{1.2}\left[\text{A} + \text{B}x^a + \text{C}x^b + \text{\&c.}\right] x' + \frac{1}{3.4}$$

$$\left[a\text{B}x^{a-1} + b\text{C}x^{b-1} + \text{\&c.}\right] x'^2 - \frac{1}{1.2.3.4.5.6}$$

$$\left[a\,(a-1)\,(a-2)\,\text{B}x^{a-3} + b\,(b-1)\,(b-2)\,\text{C}x^{b-3} + \text{\&c.}\right] x'^4 +, \text{\&c.}$$

If, see p. 268, $(\text{B}x^a + \text{C}x^b + \text{\&c.})\,x'$ is added to the right member of this equation, the sum will express the circumscribed rectangles, and we shall have

$$(y'' + y''' + \dots + y^{n+1})\,x' = \text{A}x + \frac{\text{B}x^{a+1}}{a+1} + \frac{\text{C}x^{b+1}}{b+1} + \text{\&c.}$$

$$- \frac{\text{A}x'}{2} + \frac{1}{1.2}\left[\text{A} + \text{B}x^a + \text{C}x^b + \text{\&c.}\right]$$

$$x' + \frac{1}{3.4}\left[a\text{B}x^{a-1} + b\text{C}x^{b-1} + \text{\&c.}\right] x'^2 - \frac{1}{1.2.3.4.5.6}$$

$$\left[a\,(a-1)\,(a-2)\,\text{B}x^{a-3} + b\,(b-1)\,(b-2)\,\text{C}x^{b-3} + \text{\&c.}\right] x'^4 +, \text{\&c.}$$

It is easy to perceive that the part

$$\text{A}x + \frac{\text{B}x^{a+1}}{a+1} + \frac{\text{C}x^{b+1}}{b+1} +, \text{\&c.},$$

of the inscribed an l circumscribed rectangles, which is independent of x', or does not depend on the number of equal parts into which $04 = x$ is supposed to be divided, must express the area of the curve bounded by the ordinates $0a$ and $4e$, and the intercepted parts 04 and ae of the line of the abscissæ and curve, as required; it is also evident that the terms in the rectangles, which involve x' and its powers as factors, must depend on the number of equal parts into which $04 = x$ is supposed to be divided.

From $$Anx' + \frac{Bx^a nx'}{a+1} + \frac{Cx^b nx'}{b+1} +, \&c.,$$

which is the first form of

$$Ax + \frac{Bx^{a+1}}{a+1} + \frac{Cx^{b+1}}{b+1} +, \&c.,$$

it is clear that

$$(A + Bx^a + Cx^b + \&c.)\,x' = f(x)\,x' = yx'$$

is equivalent to the differential of the curvilinear area $04ea$, and may be expressed by writing dx for x'; noticing that the x' here used need not be the same as the x' in the other terms of the rectangles described above. Also, multiplying by n, and putting $nx' = ndx = x$, which gives

$$Ax + \frac{Bx^{a+1}}{a+1} + \frac{Cx^{b+1}}{b+1} +, \&c.,$$

is clearly the same as the integral of the preceding differential, since the results are found by measuring the index of x in each term of the differential by unity or 1, and dividing by the measured index of x, which is in conformity to the common rule for finding the integral of the differential of a power.

It is hence clear that the Differential and Integral Calculus are deducible from what has been done, without using infinitesimals or limiting ratios. [See (17) at p. 44.]

It is hence easy to perceive in what sense the Integral

Calculus may be regarded as being the reverse of the Differential Calculus, and *vice versa.*

Representing $A + Bx^a + Cx^b +$, &c., by y, the expression for the sum of the inscribed rectangle, becomes

$$(y' + y'' + \ldots . + y^n)x' =$$

$$\int y dx + \frac{y'x'}{1.2} - \frac{yx'}{1.2} + \frac{1}{3.4}\frac{dy}{dx}x'^2 - \frac{1}{1.2.3.4.5.6}\frac{d^3y}{dx^3}x'^4 +, \&c.;$$

since

$$aBx^{a-1} + bCx^{b-1} + \&c. = \frac{d\,(A + Bx^a + \&c.)}{dx} = \frac{dy}{dx},$$

and

$$[a\,(a-1)\,(a-2)\,Bx^{a-3} + b\,(b-1)\,(b-2)\,Cx^{b-3} + \&c.] =$$

$$d^3\,[A + Bx^a + Cx^b + \&c.] \div dx^3 = \frac{d^3y}{dx^3},$$

and so on; a form which is substantially the same as given by Lacroix, at p. 107 of his work, from very different principles.

By adding $(y-y')\,x'$ to the right member of the preceding equation, we shall have $(y'' + y''' + \ldots . + y)\,x'$, the sum of the circumscribed rectangles, expressed by

$$\int y dx - \frac{y'x'}{1.2} + \frac{yx'}{1.2} + \frac{1}{3.4}\frac{dy}{dx}x'^2 - \frac{1}{1.2.3.4.5.6}\frac{d^3y}{dx^3}x'^4 +, \&c.$$

It may be added, that in the inscribed rectangles y' is the first ordinate and y^n the last, while in the circumscribed rectangles y'' and y^{n+1} are the first and last ordinates.

If the preceding equations are divided by x', and Σy is used to express the sum of the ordinates, taken according to the preceding directions, we shall have

$$\Sigma y = \frac{\int y dx}{x'} + \frac{y'}{1.2} - \frac{y}{1.2} + \frac{1}{3.4}\frac{dy}{dx}x' - \frac{1}{1.2.3.4.5.6}\frac{d^3y}{dx^3}x'^3 +, \&c.,$$

and

$$\Sigma y = \frac{\int y dx}{x'} - \frac{y'}{1.2} + \frac{y}{1.2} + \frac{1}{3.4}\frac{dy}{dx}x' - \frac{1}{1.2.3.4.5.6}x'^3 +, \&c.:$$

if we add y to the members of the first of these equations and y' to those of the second, and write S before y to signify the sum of all the ordinates, then the two equations concur in giving

$$Sy = \Sigma y + y$$

$$= \frac{\int y dx}{x'} + \frac{y' + y}{2} + \frac{1}{3.4}\frac{dy}{dx}x' - \frac{1}{1.2.3.4.5.6}\frac{d^3y}{dx^3}x^3 +, \&c.$$

This formula enables us to find the exact or approximate values of series, whose terms follow a given law of formation, and are equidistant from each other, or have equal intervals between them.

Thus, to find the sum of the series $1^2 + 2^2 + 3^2 + \ldots. + x^2$, we have $y = x^2$, called the general term of the series.

Hence, $\int y dx = \frac{x^3}{3} + C$, $\frac{dy}{dx} = 2x$, and $\frac{d^3y}{dx^3} = 0$,

and since the difference of the successive terms of the series 0, 1, 2, 3, &c., equals 1, we put 1 for x', and Σy becomes $\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{2.3} + C$, the arbitrary constant C being $= 0$ since the value of y', which corresponds to 0 in the series 0, 1, 2, 3, &c., is equal to 0; consequently, Σy is reduced to

$$\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{2.3},$$

to which, adding $y = x^2$, we have

$$Sy = \Sigma y + y = \frac{x^3}{3} + \frac{x^2}{2} + \frac{x}{2.3}$$

$$= \frac{2x^3 + 3x^2 + x}{6} = \frac{x\ (x+1)\ (2x+1)}{6},$$

for the sum of x terms of the proposed series. In like manner, to find the sum of the series 1, 2^3, $3^3 \ldots . x^3$, we have

$y' = 0$ and $y = x^3$, and thence

$$\int y dx = \frac{x^4}{4} + C, \text{ and } \frac{dy}{dx} = 3x^2, \quad \frac{d^3y}{dx^3} = 1.2.3, \quad \frac{d^4y}{dx^4} = 0, \text{ \&c.}$$

Hence, the formula

$$Sy = \Sigma y + y$$

$$= \int \frac{ydx}{x'} + \frac{y'+y}{2} + \frac{1}{3.4}\frac{dy}{dx}x' - \frac{1}{1.2.3.4.5.6}\frac{d^3y}{dx^3}x'^3 +, \text{ \&c.},$$

since $x' = 1$, gives $Sy = \frac{x^4}{4} + \frac{x^3}{2} + \frac{x^2}{4} - \frac{1}{120} + C$,

in which C is the arbitrary constant. To determine C, since $Sy=0$ when $x = 0$, by putting $x = 0$ we get $0 = -\frac{1}{120} + C$, which gives $C = \frac{1}{120}$; consequently, we shall have

$$Sy = \frac{x^2(x+1)^2}{4},$$

for the sum of x terms of the proposed series.

(10.) It clearly follows, from what has been done, that the differential of a function of a single variable as x, being of the form $f(x)\,dx$, by putting $fx = y$ becomes $f(x)\,dx = ydx$; which may, if we please, represent the differential of the area of a plane curve, whose ordinate corresponding to the abscissa x, is represented by $y = f(x)$.

Thus (see the fig. at p. 267), if $3d = y = f(x)$ and $3, 4 = dx$, the product $ydx = f(x)\,dx =$ the area of the rectangle $3d$D4, which may represent the differential of the curvilinear area to the right of $3d$; consequently, the area to the right of the ordinate $3d$, is the integral of the differential, supposing it to commence at the point where the curve cuts the axis of x.

If $ydx = f(x)\,dx$ is the differential of some known function of x, the integral $\int f(x)\,dx$ can be immediately found;

but if $f(x)\,dx$ can not readily be reduced to the differential of a known function of x, then $\int f(x)\,dx = \int y\,dx$, being reduced to the integral of the differential of the area of a curve, will, from the sum of the inscribed rectangles, given at p. 272, become (after a slight reduction)

$$\int y\,dx = (y' + y'' + \ldots + y^n)\,x' - \frac{y'x'}{1.2} + \frac{yx'}{1.2} - \frac{1}{3.4}\frac{dy}{dx}x'^2 + \frac{1}{1.2.3.4.5.6}\frac{d^3y}{dx^3}x'^4 -, \&c.;$$

a formula that will enable us to find an approximate value of the proposed integral, when x' is sufficiently small, particularly when taken in connection with the integral

$$\int y\,dx = (y'' + y''' + \ldots + y \text{ or } y^{n+1})\,x' + \frac{y'x'}{1.2} - \frac{yx'}{1.2} - \frac{1}{3.4}\frac{dy}{dx}x'^2 + \frac{1}{1.2.3.4.5.6}\frac{d^3y}{dx^3}x'^4 -, \&c.,$$

deduced from the formula, given at p. 272, for the sum of the circumscribed rectangles.

To illustrate what is here said, we will show how to find the integral $\int \frac{dx}{1+x^2}$ by the first of the preceding formulas, when taken from the limit $x = 0$ to the limit $x = 1$, or between the limits expressed by $\int_0^1 \frac{dx}{1+x^2}$. [See (7.) at p. 264.]

Here $y = \frac{1}{1+x^2}$, which, by putting $x = 0$, gives

$$y' = 1;$$

and putting $x = 0.1$ or $x^2 = 0.01$, gives

$$y'' = \frac{1}{1.01} = 0.990099+;$$

$x = 0.2$, gives $\qquad y''' = \dfrac{1}{1.04} = 0.961538+;$

and so on, to $x = 0.9$, which gives

$$y^{10} = 0.552486+.$$

By adding the ordinates, we have

$$y' + y'' + \ldots\ldots + y^{10} = 8.0998+;$$

and since the ordinates are drawn at intervals of 0.1, we have $x' = 0.1$, and hence get

$$(y' + y'' + \ldots\ldots + y^{10})\, x' = 8.0998 \times 0.1 = 0.80998+.$$

Also, $\qquad -\dfrac{y'x'}{1.2} = -\, 0.05 \quad$ and $\quad \dfrac{yx'}{1.2} = 0.025,$

since for y we must put $\dfrac{1}{1+2} = \dfrac{1}{2}$, the last value of y.

And we have

$$\frac{dy}{dx} = d.\frac{1}{1+x^2} \div dx = -\frac{2x}{(1+x^2)^2},$$

which, by putting 1 (the last value of x) for x, gives

$$\frac{dy}{dx} = -\frac{1}{2};$$

consequently, $\qquad -\dfrac{1}{3.4}\dfrac{dy}{dx}\, x'^2 = 0.000416+.$

Hence, rejecting the remaining terms, on account of their comparative minuteness, and adding the terms found, we have

$$\int_0^1 \frac{dx}{1+x^2} = 0.80998 - 0.05 + 0.025 + 0.000416 = 0.78539+.$$

From the third form of the table given at p. 257, by putting $a = 1$ and $b = 1$, it is clear that $\int_0^1 \dfrac{dx}{1+x^2}$ equals the length of an arc of 45° of the circumference of a circle whose radius $=1$, which is well known to be $0.78539+$; consequently, the arc has been correctly found to five decimal places, by a calculation of remarkable simplicity. By drawing the ordinates sufficiently near each other, it is clear that we may in this way find the circumference correctly to any finite number of decimal places.

For a curve such, that the differential of its area is that of an integral of a known form, we will show how to find the area of a parabola.

Thus, let $ax = y^2$ represent the equation of the parabola; then, by taking the differentials, we have $dx = \frac{2ydy}{a}$, which gives $ydx = \frac{2y^2dy}{a}$, whose integral is

$$\int ydx = \frac{2}{a}\int y^2dy = \frac{2}{3}\frac{y^3}{a} + C = \frac{2}{3}xy + C,$$

from the equation of the curve.

To find the constant C, we shall suppose the area to commence at the vertex of the curve; then, $x = 0$ gives $\int ydx = 0$, and, of course, we shall have $C = 0$, and the area becomes $\int ydx = \frac{2}{3}xy =$ *two-thirds of the semi-parabola's circumscribing rectangle;* agreeably to a *well-known property of the parabola.*

(11.) Resuming the figure at p. 267, and supposing it to revolve about the axis of x or 04, it is manifest that the curvilinear area will describe a portion of a solid of revolution; and that the inscribed rectangles will describe cylinders inscribed within the solid, while the circumscribed rectangles will describe cylinders circumscribing the solid, such that the solid will be greater than the sum of all the inscribed cylinders, and less than the sum of all the circumscribed cylinders.

If $\pi = 3.14159$, &c., the cylinder generated by the revolution of the rectangle $01\text{A}a$ will, by mensuration, be expressed by $\pi y'^2 x'$, and in like manner all the remaining inscribed cylinders may be expressed.

Hence, if y, y', y'', &c., are changed into πy^2, $\pi y'^2$, $\pi y''^2$, &c., the formula for the sum of the inscribed rectangles, at p. 275, will become

$$\int \pi y^2 dx = \pi \int y^2 dx = \pi \left[(y'^2 + y''^2 + \ldots + y^{n2})\, x' - \frac{y'^2 x'}{1.2} + \frac{y^2 x'}{1.2} - \frac{1}{3.4}\,\frac{dy^2}{dx} + \frac{1}{1.2.3.4.5.6}\,\frac{d^3 y^2}{dx^3}\, x'^4 + \&c. \right],$$

the formula for the sum of all the cylinders inscribed in the portion of the solid of revolution; and in much the same way, the sum of all the cylinders which circumscribe the solid may also be found.

Noticing, that this process will be unnecessary when the integral expressed by $\int y^2 dx$ can readily be found.

Thus, in finding the contents of the paraboloid described by the revolution of the parabola $ax = y^2$ about the axis of x.

Since $\quad dx = \dfrac{2y dy}{a}, \quad$ we have $\quad y^2 dx = \dfrac{2y^3 dy}{a},$

and thence we get

$$\pi \int y^2 dx = \pi \int \frac{4y^3 dy}{2a} = \frac{\pi y^4}{2a} = \frac{\pi y^2 x}{2};$$

which equals half of the cylinder which circumscribes the paraboloid; noticing, that no constant is necessary, since the paraboloid equals naught when $x = 0$.

For another example, we will show how to find the contents (*or cubature*) of a sphere whose radius equals R.

From what is shown at pp. 210 and 211, it is manifest that if the sphere is cut by a plane whose perpendicular distance from the center is x, the section will be a circle, such that $R^2 - x^2$ will equal the square of the radius of the section; consequently,

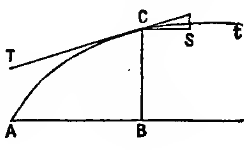
$$\pi (R^2 - x^2)\, dx = \pi (R^2 dx - x^2 dx)$$

is clearly the differential of the portion of the sphere, between the cutting plane and a parallel plane passing through the center of the sphere. Hence, by taking the integral from $x = 0$ to $x = R$, we have

$$\pi \int_0^R (R^2 - x^2)\, dx = \pi \left(R^3 - \frac{R^3}{3}\right) = \frac{2\pi}{3} R^3$$

for half the sphere; consequently, the contents of the whole sphere is $\frac{4\pi}{3} R^3$, which clearly equals two-thirds of the circumscribing cylinder.

(12.) We now propose to show how to find the lengths of plane curves.



Thus, let AB and BC represent the abscissa and ordinate of any plane curve AC, having A for its vertex, which we shall take for the origin of the co-ordinates, supposed to be rectangular. Then, representing the arc of the curve AC by z, the abscissa AB and ordinate BC by x and y, we may clearly take the very short line Cs parallel to AB, to stand for dx, the differential of x, and st parallel to BC or y, meeting the tangent to the curve at C in t, to stand for dy, then it is clear that Ct, the hypotenuse of the right triangle Cst, must equal dz, the differential of z, or we shall have

$$dz = \sqrt{(dx^2 + dy^2)} = dx \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}$$

for the differential of the arc AC $= z$.

What is here affirmed, is clear from the definition of a tangent given at p. 125, which is that the differential coefficient $\frac{dy}{dx}$ at the point C in the curve must be the same as in the tangent; consequently, using Cs to represent dx, st must represent dy, and thence Ct must clearly represent dz, as above.

Because the approximate method of finding the integral of the differential is sufficiently evident from what has heretofore been done, we shall not stop to give it.

Thus, to find the length of the curve whose equation is $y^3 = ax^2$, called the equation of the semicubical parabola, by taking the differentials, we readily get

$$dy = \frac{2}{3} a^{\frac{1}{3}} x^{-\frac{1}{3}} dx,$$

and thence $$dz = \left(x^{\frac{2}{3}} + \frac{4}{9} a^{\frac{2}{3}}\right)^{\frac{1}{2}} x^{-\frac{1}{3}} dx;$$

whose integral is $$z = \left(x^{\frac{2}{3}} + \frac{4}{9} a^{\frac{2}{3}}\right)^{\frac{3}{2}} + \mathrm{C},$$

C being the arbitrary constant. If x and z equal naught at the origin of the co-ordinates, we shall have

$$\left(\frac{4}{9} a^{\frac{2}{3}}\right)^{\frac{3}{2}} + \mathrm{C} = 0, \quad \text{or} \quad \mathrm{C} = -\frac{8}{27} a.$$

Hence the correct integral becomes

$$z = \left(x^{\frac{2}{3}} + \frac{4}{9}a^{\frac{2}{3}}\right)^{\frac{3}{2}} - \frac{8}{27}a;$$

consequently, the proposed curve is said to be *exactly rectifiable*, because the integral of its differential can be exactly found.

For another example, we will find the length of the common parabola, its equation being $ax = y^2$.

By taking the differentials of its members, we have

$$adx = 2ydy, \quad \text{or} \quad dx = \frac{ydy}{\frac{a}{2}},$$

which, by putting $\frac{a}{2} = b$, gives $dx^2 = \frac{y^2dy^2}{b^2}$; consequently,

$$dx^2 + dy^2 = \frac{b^2 + y^2}{b^2}dy^2,$$

or

$$\sqrt{(dx^2 + dy^2)} = dz = \frac{\sqrt{(b^2 + y^2)}\,dy}{b}$$

$$= \frac{bdy}{\sqrt{(b^2 + y^2)}} + \frac{y^2dy}{b\sqrt{(b^2 + y^2)}}.$$

Hence, since

$$\frac{bdy}{\sqrt{(b^2 + y^2)}} = \left(1 + \frac{y}{\sqrt{(y^2 + b^2)}}\right)bdy \div [y + \sqrt{(y^2 + b^2)}],$$

and that

$$\frac{y^2dy}{b\sqrt{(y^2 + b^2)}} = -\frac{bdy}{2\sqrt{(b^2 + y^2)}} + \frac{1}{2b}d.(b^2y^2 + y^4)^{\frac{1}{2}},$$

we have reduced $\frac{\sqrt{(b^2 + y^2)}\,dy}{b}$ to

$$\frac{1}{2b}d.\sqrt{(b^2y^2 + y^4)} + \frac{b}{2}\left(1 + \frac{y}{\sqrt{(y^2 + b^2)}}\right)$$

$$dy \div [y + \sqrt{(y^2 + b^2)}].$$

Hence (see the last example at p. 256), we shall have

$$\int \frac{\sqrt{(b^2+y^2)}\,dy}{b} = \frac{\sqrt{(b^2y^2+y^4)}}{2b} + \frac{b}{2}\log[y+\sqrt{(y^2+b^2)}]+C$$

$$= \frac{y}{2b}\sqrt{(y^2+b^2)} + \frac{b}{2}\log[y+\sqrt{(y^2+b^2)}] + C,$$

by representing hyperbolic logarithms by log, and using C to represent the arbitrary constant.

By putting $y = 0$, we shall clearly have $z = 0$, and thence $C = -\frac{b}{2}\log b$; which reduces the integral to

$$\int \frac{\sqrt{(b^2+y^2)}\,dy}{b} = \frac{y}{2b}\sqrt{(y^2+b^2)} + \frac{b}{2}\log\frac{y+\sqrt{(y^2+b^2)}}{b}.$$

Hence the common parabola is rectifiable in algebraic and transcendental terms, but not in algebraic quantities, like the preceding example.

For another example, it may be proposed to find an arc of a cycloid, reckoned from its vertex.

By referring to page 150, we have $dy = \sqrt{\frac{2r-x}{x}}\,dx$, r being the radius of the generating circle, and x and y the abscissa and ordinate; consequently,

$$dy^2 + dx^2 = dz^2 = \frac{2r}{x}\,dx^2,$$

or, $$\int dz = \int \frac{\sqrt{8r}}{2}\frac{dx}{\sqrt{x}} = 2\sqrt{2r}\sqrt{x} = 2\sqrt{2rx},$$

which needs no correction, supposing the integral to commence with x.

Hence, see the fig. at p. 149, it is clear that the cycloidal arc DG = 2DF = twice the chord of the corresponding arc of the generating circle; consequently, DG^2 or $z^2 = 8rx$.

(13.) We will now proceed to show how to find the surfaces of solids of revolution.

Thus, supposing the fig. in (12), at p. 279, revolves about its axis AB, it will generate what is called a solid of revolution, whose arc AC will describe its curve surface, which we propose to show how to find.

Because $Ct = dz =$ the differential of $AC = z$, it is manifest that dz, multiplied by the circumference of the circle whose radius equals $BC = y$, will represent the differential of the surface described by the arc AC in one revolution about its axis AB. Hence, putting $\pi = 3.14159+$, and representing the surface described by AC by S, we shall clearly have $dS = 2\pi y dz$ for the differential of the described surface.

Thus, to find the surface of a sphere whose radius is r, we shall evidently have $r : y :: dz : dx$ (or CS), (from similarity of triangles, since the radius drawn to C cuts Ct perpendicularly, and that when the angle TCB is acute, the center is at the right of B in AB), or $ydz = rdx$; consequently, $dS = 2\pi y dz$ reduces, by substitution, to $dS = 2\pi r dx$, whose integral is $\int dS = \int 2\pi r dx$ or $S = 2\pi r x$, which needs no correction, supposing the surface S to commence with x. If for x we put $2r$, the integral becomes $S' = 4\pi r^2$, where S' stands for the whole surface; consequently, since $\pi r^2 =$ the surface of a great circle of the sphere, it follows that S', the whole surface, *equals four times the area of a great circle of the sphere;* and from $S = 2\pi r x$, it is manifest that the variations of S are proportional to those of x.

If we take dz, in the parabola given at p. 282, we shall have

$$dS = \frac{2\pi\sqrt{(b^2 + y^2)}}{b} y dy,$$

for the differential of the surface of the common paraboloid, whose integral

$$\int dS = \frac{2\pi}{3}\int \sqrt{(b^2+y^2)}\,ydy \quad \text{gives} \quad S = \frac{4\pi}{3b}(b^2+y^2)^{\frac{3}{2}} + C,$$

C being the arbitrary constant.

To determine C, we suppose S and y to commence together, and thence get

$$C = -\frac{4\pi}{3}b^2, \quad \text{which gives} \quad S = \frac{4\pi}{3b}\left[(b^2+y^2)^{\frac{3}{2}} - b^3\right]$$

for the correct integral.

We will now show how to find the area of the surface generated by the revolution of the catenarian curve about its axis, supposing the equation between the length of the curve and the corresponding abscissa to be expressed by the equation $z^2 = 2ax + x^2$, or by its equivalent, $\sqrt{(a^2+z^2)} = a + x$.

By taking the differentials, we have

$$dx = \frac{zdz}{\sqrt{(a^2+z^2}};$$

consequently, since $dz^2 = dx^2 + dy^2$, we have

$$dy^2 = dz^2 - dx^2 = dz^2 - \frac{z^2dz^2}{a^2+z^2} = \frac{a^2dz^2}{a^2+z^2}$$

or $$dy = \frac{adz}{\sqrt{(a^2+z^2)}}.$$

Because

$$dS = 2\pi ydz = 2\pi(ydz + zdy - zdy) = 2\pi(dyz - zdy),$$

we have, by taking the integral,

$$S = 2\pi\left(yz - a\int\frac{zdz}{\sqrt{(a^2+z^2)}}\right) = 2\pi\left[yz - a\sqrt{(a^2+z^2)}\right] + C,$$

C being the arbitrary constant, which equals $2\pi a^2$ or

$$S = 2\pi\left[yz + a^2 - a\sqrt{(a^2+z^2)}\right],$$

when S = 0 at the vertex of the curve. Because

$$a^2 - a\sqrt{(a^2 + z^2)} = -ax,$$

our equation is equivalent to $S = 2\pi(yz - ax)$, as required.

For the last example, we will show how to find the surface generated by the revolution of a cycloid around its base.

Thus, by referring to the fig. at p. 149, since BD $= 2r$ and DE $= x$, we have BE $= 2r - x =$ the perpendicular from G to the base AC of the cycloid, and which revolves about the base; and, from the example at p. 282, we have

$$dz = \sqrt{\frac{2r}{x}}\,dx = \sqrt{2r}\,x^{-\frac{1}{2}}dx.$$

Hence, putting $2r - x$ for y, and $\sqrt{2r}\,x^{-\frac{1}{2}}dx$ for dz in $dS = 2\pi y dz$, we shall have

$$dS = 2\pi\sqrt{2r}\,(2r - x)\,x^{-\frac{1}{2}}dx = 2\pi\sqrt{2r}\,(2rx^{-\frac{1}{2}}dx - x^{\frac{1}{2}}dx)$$

for the differential of the surface generated by the revolution of the cyclodial arc DG about BC, since this increases positively, while that described by GC decreases. By taking the integrals, we have

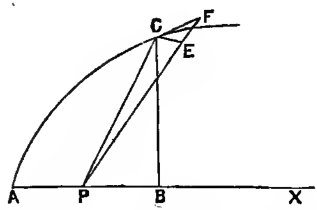
$$S = 2\pi\sqrt{2r}\left(\int 2rx^{-\frac{1}{2}}dx - \int x^{-\frac{1}{2}}dx\right) = 2\pi\sqrt{2r}\left(4rx^{\frac{1}{2}} - \frac{2}{3}x^{\frac{3}{2}}\right);$$

which needs no correction, supposing the integral to commence with x. By putting $2r$ for x, we have

$$S = 2\pi\sqrt{2r}\left(4r\sqrt{2r} - \frac{4}{3}r\sqrt{2r}\right) = 2\pi\sqrt{2r} \times \frac{8}{3}r\sqrt{2r} = \frac{32\pi r^2}{3}$$

for the surface described by the semicycloidal arc DC about BC, and of course $\frac{64\pi r^2}{3}$ is the whole surface described by the revolution of the cycloid around its base, as required.

(14.) We will now show how to use polar co-ordinates in finding the areas and lengths of curves.



Thus, let AC be a curve, having P for its pole, PC $= r$ and the angle APC $= \phi$ for the polar co-ordinates of any point C of the curve; then, shall $\frac{r^2 d\phi}{2}$ equal the differential of the curvilinear area APC.

For, taking PB and the perpendicular BC for the rectangular co-ordinates of C, and denoting them by x and y, their origin being at P; then, from what has been shown, ydx is the differential of the area ABC. (See p. 266, &c.)

Also, since the area of the triangle PBC equals $\frac{xy}{2}$, and that the curvilinear area APC = the area ABC − triangle PBC = the area ABC $- \frac{xy}{2}$, by taking the differentials of those equals, we shall have the differential of the curvilinear area

$$\text{APC} = ydx - \frac{ydx + xdy}{2} = \frac{ydx - xdy}{2}.$$

Because $\quad$ tan angle BPC $= -\tan\phi = \frac{y}{x}$,

by taking the differentials of these equals, we shall have

$$-\frac{d\phi}{\cos^2\phi} = \frac{xdy - ydx}{x^2}, \quad \text{or} \quad ydx - xdy = \frac{x^2 d\phi}{\cos^2\phi} = r^2 d\phi;$$

consequently, we have the differential of the curvilinear area

$APC = \frac{r^2 d\phi}{2}$, as required. Hence, if PE makes the small angle CPE equal to $d\phi$ with CP, $d\phi$ being an arc of a circle to radius 1; then, it is clear that the circular sector CPE, whose center is at P, will represent the differential of the area APC.

Thus, if AC is a parabola, having Ax for its axis and P for its focus; then, representing AP by m, $4m$ will be its parameter, and we shall have (by a well-known property of the curve)

$$4m\,(m + x) = 4m^2 + 4mx = y^2,$$

or $$4m^2 + 4mx + x^2 = (2m + x)^2 = x^2 + y^2 = r^2,$$

which gives $r - x = 2m$; or, since $x = -r\cos\phi$, we have $r\,(1 + \cos\phi) = 2m$; and, since

$$1 + \cos\phi = 2\cos^2\frac{\phi}{2}, \quad \text{we get} \quad r = \frac{m}{\cos^2\frac{\phi}{2}},$$

for the polar equation of the parabola.

Hence, $\frac{r^2 d\phi}{2}$ becomes

$$\frac{m^2\frac{d\phi}{2}}{\cos^4\frac{\phi}{2}} = \frac{m^2 d\,.\tan\frac{\phi}{2}}{\cos^2\frac{\phi}{2}} = m^2\left(d\,.\tan\frac{\phi}{2} + \tan^2\frac{\phi}{2}\,d\,.\tan\frac{\phi}{2}\right),$$

whose integral taken from $\phi = 0$, gives

$$\int\frac{r^2 d\phi}{2} = m^2\left(\tan\frac{\phi}{2} + \frac{1}{3}\tan^3\frac{\phi}{2}\right),$$

as required; a result that is very important in treating of the parabolic motion of comets. (See Vince's "Astronomy," vol. I., p. 428.)

For another example, we will find the area of the spiral of

Archimedes, whose equation is $r = a\phi$. By taking the differentials, we have

$$d\phi = \frac{dr}{a}, \quad \text{which gives} \quad \frac{r^2 d\phi}{2} = \frac{r^2 dr}{2a};$$

whose integral, taken from $r = 0$, is

$$\int \frac{r^2 d\phi}{2} = \frac{r^3}{6a}.$$

In like manner, from the equation $r = a^\phi$, the equation of the logarithmic spiral, by taking the hyperbolic logarithms, we have $\log r = \phi \log a$, whose differentials give

$$\frac{dr}{r} = d\phi \log a \quad \text{or} \quad d\phi = \frac{dr}{r \log a},$$

and thence
$$\frac{r^2 d\phi}{2} = \frac{r dr}{2 \log a};$$

whose integral, taken from $r = 0$, gives

$$\int \frac{r^2 d\phi}{2} = \frac{r^2}{4 \log a}.$$

By taking $r = \frac{a}{\phi}$, or $\phi = \frac{a}{r}$, the equation of the hyperbolical spiral, we get

$$d\phi = -\frac{adr}{r^2}, \quad \text{and thence} \quad \frac{r^2 d\phi}{2} = -\frac{adr}{2};$$

whose integral, taken from $r = r'$, is

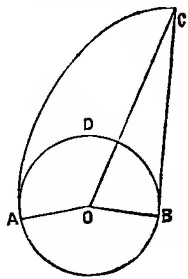
$$\int \frac{r^2 d\phi}{2} = \frac{a(r' - r)}{2},$$

which, taken to $r = 0$, or an infinitesimal, is

$$\int \frac{r^2 d\phi}{2} = \frac{ar'}{2},$$

which equals the area of a right-angled triangle, whose perpendicular sides are a and r'.

REMARKS.—In treating of spirals, it will sometimes be convenient to consider the pole as moving, according to some given law, instead of being fixed, as is usually done.



Thus, let a thread be wound from A around the circle ADB in the direction of the letters A, D, and B; then, when the thread is unwound from A, so as to be constantly a tangent to the circle, the extremity A of the thread will describe a curve AC, called the *Involute of the Circle*, to which the thread is clearly constantly perpendicular, while it unwinds; so that BC denoting any unwound part of the thread, it is manifest that BC cuts the curve AC perpendicularly at C, and is at the same time a tangent to the circle at B, and equal in length to the circular arc ADB. (See Sec. VI., p. 163.)

We now propose to show how to find the area of the involute bounded by the arc AC, the unwound part CB of the thread, and the circular arc ADB.

Representing OC by r, the radius OB of the circle by R, the right triangle BCO gives BC $= \sqrt{(r^2 - R^2)} =$ the circular arc ADB. Hence, $\frac{\sqrt{(r^2 - R^2)}}{R}$ equals the arc to radius $= 1$, which represents the angle AOB, which we shall take for ϕ, and for r we shall take $\sqrt{(r^2 - R^2)}$.

Hence, from $\phi = \frac{\sqrt{(r^2 - R^2)}}{R}$, we get $d\phi = \frac{rdr}{R\sqrt{(r^2 - R^2)}}$; and this multiplied by $r^2 - R^2$, the square of the corresponding radius vector, gives

$$\frac{(r^2 - R^2)\,d\phi}{2} = \frac{\sqrt{(r^2 - R^2)}\,rdr}{2R}$$

for the differential of the sought area: since the angular motion of BC is clearly the same as that of the perpendicular radius OB, it is clear that the angle ϕ has been correctly represented, while the pole moves from A, on the circular arc from A through D to B.

By taking the integral of the differential equation, we have

$$\int \frac{\sqrt{(r^2 - R^2)}\,rdr}{2R} = \frac{(r^2 - R^2)^{\frac{3}{2}}}{6R},$$

for the correct area; supposing it to commence when $r = R$, or when $\sqrt{(r^2 - R^2)} = 0$.

Hence, since the circular sector (from the principles of geometry) ADB = OBC, it follows that we shall have the area

$$\text{AOBC} - \text{OBC} = \text{the area AOC} = \text{the area ACBD} = \frac{(r^2 - R^2)^{\frac{3}{2}}}{6R},$$

which agrees with the area usually found. (See p. 76 of Vince's "Fluxions.")

(15.) To find the lengths of curves by using polar coordinates, we proceed as follows:

Thus, by using the figure and notation in (14), at p. 285, we have $\quad x = -r\cos\phi \quad$ and $\quad y = r\sin\phi$,

which give $\quad dx = -\cos\phi dr + r\sin\phi d\phi$

and $\quad dy = \sin\phi dr + r\cos\phi d\phi$;

which, by taking the square roots of the sums of their squares, gives

$$\sqrt{(dx^2 + dy^2)} = \sqrt{r^2 d\phi^2 + dr^2} = \sqrt{\left(r^2 + \frac{dr^2}{d\phi^2}\right)} \times d\phi.$$

From what is shown at pp. 133 and 134, since

$\sqrt{(dx^2 + dy^2)} = dz$, the differential of the arc AC,

it results that in polar co-ordinates we shall have

$$dz = \sqrt{(r^2 d\phi^2 + dr^2)} = \sqrt{\left(r^2 + \frac{dr^2}{d\phi^2}\right)}\, d\phi$$

for the differential of the arc $AC = z$, as required.

REMARKS.—1. By referring to the figure at p. 131, and to what has there been done, taken in connection with what has been done above, it follows that the normal to the curve at C, limited by the perpendicular through P to the radius vector $PC = r$, equals $\sqrt{\left(r^2 + \frac{dr^2}{d\phi^2}\right)}$, since (see page 132) $\frac{dr^2}{d\phi^2}$ = the square of the subnormal. Hence, by putting the normal $\sqrt{\left(r^2 + \frac{dr^2}{d\phi^2}\right)} = N$, we have $Nd\phi$, from what is shown above, for the differential of the curve z, in polar co-ordinates; in which ϕ = the angle APC, and observing that $d\phi$ equals the differential of the angle which the perpendicular to PC through P makes with the axis AB of x; noticing, that (if we please) we may regard $-d\phi$ as being the differential of the angle which the normal to the curve at C makes with the perpendicular through P to the radius vector $r = PC$.

2. If a perpendicular from the pole P is drawn to the tangent CF produced, and t denotes the distance of its intersection from C, then, from equiangular triangles, we shall

have $t : \text{PC} :: \text{FE} : \text{CF}$, or $t : r :: dr : dz$,

which gives $dz = \frac{rdr}{t}$ for another expression of the differential in polar co-ordinates.

Otherwise.—Referring to the figure at p. 131, it is clear that the right triangle rSN gives the radius vector

$$r\text{S} = r = \text{SN} \sin \text{N} = \text{N}' \sin \text{N},$$

by using N′ to represent the normal SN.

By taking the differentials of the equation $r = \text{N}' \sin \text{N}$, supposing N′ to be constant or invariable, we have

$$dr = \text{N}' \cos \text{N} d\text{N} = \text{N}' \cos \text{N} d\phi,$$

as is manifest from what has been shown; also, from what has been shown, we have $\text{N}'d\phi = dz$, the differential of the arc AS, and of course $dr = \cos \text{N} dz$. Since $\cos \text{N} = \sin r\text{SN}$, if we multiply the members of this by r, we shall have $rdr = r \sin r\text{SN} dz$, in which $r \sin r\text{SN} =$ the perpendicular from r to SN, which evidently equals t.

Hence, we shall have $tdz = rdr$, or $dz = \frac{rdr}{t}$, the same result as found from the preceding method.

Thus, to find the length of the logarithmic, or equiangular spiral, since $\frac{r}{t} =$ the secant of the angle at which the radius vector cuts the curve, if we represent the secant by s, we shall have $dz = sdr$, in which s is constant, since the radius vector always cuts the curve at the same angle.

Hence, by taking the integral, we shall have $\int dz = s \int dr$ or $z = sr$, which needs no correction, supposing the integral to commence with r; and it follows that z varies as r.

For another example, we will take the spiral of Archimedes, whose equation is $r = a\phi$.

By taking the differentials, we have $dr = ad\phi$; and thence

$$dz = \sqrt{(r^2 d\phi^2 + dr^2)} = \sqrt{\left(\frac{r^2}{a^2} + 1\right)}\, dr = \frac{1}{a}\sqrt{(r^2 + a^2)}\, dr,$$

which agrees in form with the differential of the length of the common parabola given at p. 281, when we put y and b for r and a. Hence, putting r and a for y and b at p. 282, we have

$$z = \frac{r}{2a}\sqrt{(r^2 + a^2)} + \frac{a}{2}\log\frac{r + \sqrt{(r^2 + a^2)}}{a}$$

as required, in which z commences with r.

For further illustration, we will find the length of the involute of the circle.

By proceeding as at pp. 288 and 289, and adopting the same notation as there used, we have

$$d\phi = \frac{rdr}{R\sqrt{(r^2 - R^2)}},$$

which, multiplied by $\sqrt{(r^2 - R^2)}$, taken for the radius vector, gives

$$dz = \sqrt{(r^2 - R^2)}\, d\phi = \frac{rdr}{R};$$

whose integral gives

$$z = \int \sqrt{(r^2 - R^2)}\, d\phi = \int \frac{rdr}{R} = \frac{r^2 - R^2}{2R},$$

supposing that the integral commences with $r = R$; noticing, that in this solution the pole is supposed to move from A, around the circumference of the circle, in the order of the letters A, D, B, as at p. 288.

For the last example, we will take the reciprocal spiral, whose equation is $r = \frac{1}{\phi}$ or $\phi = \frac{1}{r}$.

By taking the differentials we have

$$d\phi = \frac{dr}{r^2}, \quad \text{which gives} \quad d\phi^2 = \frac{dr^2}{r^4}, \quad \text{or} \quad r^2 d\phi^2 = \frac{dr^2}{r^2};$$

and thence we have

$$dz = \sqrt{(r^2 d\phi^2 + dr^2)} = \sqrt{\left(\frac{1}{r^2} + 1\right)} dr = \sqrt{(r^2+1)} \frac{dr}{r}$$

$$= \frac{(r^2+1)\dfrac{dr}{r}}{\sqrt{(r^2+1)}} = \frac{rdr}{\sqrt{(r^2+1)}} + \frac{dr}{r\sqrt{(r^2+1)}}$$

$$= d\sqrt{(r^2+1)} + d(\log r) - d\,\{\log[\sqrt{(1^2+r^2)}+1]\};$$

consequently, by taking the integrals, we shall have

$$z = \sqrt{(r^2+1)} - \log \frac{\sqrt{(r^2+1)}+1}{r} + C,$$

C being the arbitrary constant; this integral is clearly the same as $z = \sqrt{(r^2+1)} + \log \dfrac{r}{\sqrt{(r^2+1)}+1} + C,$

which will clearly enable us to find the value of z that corresponds to the interval between any finite values of r.

(16.) We will now show how to find the contents or volume of a solid, the equation of whose surface can be expressed by an equation between the rectangular co-ordinates, x, y, and z, without regarding the body as being a solid of revolution.

It is manifest that we may regard the very small parallelopiped expressed by $dxdydz$, as being the differential of the solid, and represent its integral by $\int\int\int dxdydz$, by using the $\int$ successively to represent the separate integrations with reference to z, y, and x.

Thus, by performing the first integration with reference to z taken between the plane x, y, and the surface of the body, the integral is reduced to the form $\int\int zdxdy$; which may be integrated again by regarding z as being a function of x and y.

If we at first integrate with reference to y by regarding x as constant, we shall have $\iint z dx dy = \int dx \int z dy$, in which $\int z dy$ denotes the area of a section of the solid by a plane that is parallel to the plane of the axes of z and y; then, having found the integral $\int z dy$, we can find the integral $\int dx \int z dy$, which being taken between proper limits of x, will give the required volume of the solid.

It is manifest that we may perform the integrations in the forms $\iint z dx dy = \int dy \int z dx$, instead of using the preceding forms; noticing, that those forms which are the simplest in the integrations are always to be chosen.

It ought to be added, that to find the integrals in the simplest manner, the planes of the co-ordinates should be drawn, if possible, so that they may divide the body into equal parts.

Thus, to find the volume of the ellipsoid whose equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, it is manifest that the planes of its axes are those of its co-ordinates.

Then, to simplify the equation still further, we put $x = ax'$, $y = by'$, and $z = cz'$, which, by substitution, reduce the equation to $x'^2 + y'^2 + z'^2 = 1$; the equation of the surface of a sphere, having 1 for its radius. Since

$$\int z dx = ac \int z' dx', \quad \text{and} \quad \int dy' \int z' dx' = abc \iint z' dy' dx',$$

it manifestly follows, that if we multiply the volume of the sphere whose radius equals 1, by the product abc, the result will express the volume of the proposed ellipsoid.

Now, from $$x'^2 + y'^2 + z'^2 = 1,$$
we have $$z' = \sqrt{(1 - y'^2 - x'^2)} = \sqrt{(r'^2 - x'^2)},$$
by putting $r'^2 = 1 - y'^2$; consequently, we shall have

$$\iint z'dy'dx' = \iint \sqrt{(r'^2 - x'^2)}dy'dx' = \int dy' \int \sqrt{(r'^2 - x'^2)}\, dx'.$$

It is manifest that to find the integral $\int \sqrt{(r'^2 - x'^2)}\, dx'$, r' must be regarded as constant, and that it will be sufficient to take the integral from $x' = 0$ to $x' = r'$, since the whole integral can thence be readily found.

It is manifest that

$$\int_0^{r'} \sqrt{(r'^2 - x'^2)}\, dx' = \frac{r'^2\pi}{4} = \frac{\pi}{4}(1 - y'^2),$$

which equals the fourth part of the area of a circle, whose radius is r' or $\sqrt{(1 - y^2)}$. Hence,

$$\int dy' \int (r'^2 - x'^2)\, dx' \quad \text{becomes} \quad \frac{\pi}{4}\int (1 - y'^2)\, dy',$$

whose integral it will be sufficient to take from $y' = 0$ to $y' = 1$, which gives $\frac{\pi}{4}\int_0^1 (1 - y'^2)\, dy' = \frac{\pi}{6}$, for the eighth part of the sphere, whose radius $= 1$.

Hence, $\frac{4\pi abc}{3}$ is evidently equal to the contents or volume of the proposed ellipsoid, as required.

It may be added, that, to simplify the integrals $\iint zdxdy$, we sometimes put $y = xu$, and thence, since x and y are independent variables, get $dy = xdu$, which reduces

$$\iint zdxdy \quad \text{to} \quad \iint zduxdx = \int du \int zxdx.$$

Thus, in finding the volume of the sphere whose equation is $x^2 + y^2 + z^2 = \mathrm{R}^2$, we have

$$z = \sqrt{(\mathrm{R}^2 - x^2 - y^2)} = \sqrt{(\mathrm{R}^2 - x^2 - x^2u^2)} = \sqrt{[\mathrm{R}^2 - x^2(1+u^2)]},$$

by putting $y = xu$. Hence, the integrals $\iint z\,du\,x\,dx$ become

$$\int du \int [\mathrm{R}^2 - x^2\;(1+u^2)]^{\frac{1}{2}}\,x\,dx =$$

$$\int du\left(\mathrm{C} - \frac{[\mathrm{R}^2 - x^2(1+u^2)]^{\frac{3}{2}}}{3\,(1+u^2)}\right),$$

by regarding u as constant in the integration, and using C for the arbitrary constant. Supposing the integral to commence when $x = 0$, we have $\mathrm{C} = \dfrac{\mathrm{R}^3}{3\,(1+u^2)}$; then, if the integral is extended to $x = \dfrac{\mathrm{R}}{\sqrt{(1+u^2)}}$, we shall have

$$\int du \int \sqrt{\mathrm{R}^2 - x^2(1+u^2)}\;x\,dx = \frac{\mathrm{R}^3}{3}\int \cdot \frac{du}{1+u^2}$$

$$= \frac{\mathrm{R}^3}{3}\int \frac{d\,\frac{y}{x}}{1+\left(\frac{y}{x}\right)^2} = \frac{\mathrm{R}^3}{3}\tan^{-1}\frac{y}{x},$$

in which the arc $\tan^{-1}\dfrac{y}{x}$ must clearly be taken from $\dfrac{y}{x} = 0$ to $\dfrac{y}{x} =$ infinity, and of course the arc equals $\dfrac{\pi}{2}$; consequently, $\dfrac{\mathrm{R}^3\pi}{6}$ is an eighth part of the sphere, and its volume equals $\dfrac{4\pi\mathrm{R}^3}{3}$, as required.

Remarks.—1. If we change the rectangular co-ordinates x and y into the polar co-ordinates $x = r\cos\phi$ and $y = r\sin\phi$,

r being the radius vector (in the plane x, y, drawn from the origin), and ϕ the angle it makes with the axis of x; then, by assuming $dx = -r \sin \phi d\phi$ from $x^2 + y^2 = r^2$, we have $ydy = rdr$, on account of the independence of x and y, or $dy = \frac{rdr}{y} = \frac{dr}{\sin \phi}$; consequently, $dxdy = -rdrd\phi$; noticing, that if we had assumed $dy = r \cos \phi d\phi$, we should, from $x^2 + y^2 = r^2$, have had $dx = \frac{dr}{\cos \phi}$, and thence $dxdy = rdrd\phi$; so that regarding dr and $d\phi$ as being positive, the transformations ought to be taken absolutely, or without reference to their signs. Hence, $zdxdy$ will be changed to $zrdrd\phi$; which will often be found very useful in integration. Thus, to find the volume of the sphere whose equation is $R^2 = x^2 + y^2 + z^2 = r^2 + z^2$, we immediately, on account of the constancy of R, get

$$rdr + zdz = 0, \quad \text{or} \quad rdr = -zdz,$$

which reduces $zrdrd\phi$ to $-z^2dzd\phi$. Hence, we have

$$\iint -z^2dzd\phi = -2\pi \int z^2dz,$$

since the integral with regard to ϕ ought clearly to be taken throughout the whole circumference. By taking the integral $-2\pi \int z^2dz$ from $z = R$ to $z = 0$, we have

$$-2\pi \int_R^0 z^2dz = \frac{2R^3\pi}{3}$$

for the volume of half the sphere, and of course that of the whole sphere is $\frac{4\pi R^3}{3}$; the same as found by the preceding methods.

2. It is easy to perceive that we may transform the infin-

itesimal solid $dzdydx$ by polar co-ordinates after the following manner.

Thus, let r denote its distance from the origin of the co-ordinates, and θ the angle it makes with the plane of x, y; then, $r\cos\theta$ being represented by r', it will be the projection of r on the plane x, y, and we also have $r\sin\theta = z$.

Hence, if r' makes the angle ϕ with the axis of x, we shall, from what has been previously shown, get

$$dxdy = r'dr'd\phi.$$

Since $r^2 = r'^2 + z^2$, if we assume $dz = r\cos\theta d\theta$, it results, from the independence of r' and z, that we must assume $r'dr' = rdr$; consequently, $dzdydx$ is transformed to

$$r^2\cos\theta drd\theta d\phi.$$

Hence, $$\iiint dzdydx = \int^3 dydxdz,$$

called a triple integral, is transformed to the triple integral

$$\int^3 r^2\cos\theta drd\theta d\phi = \int r^2dr\int\cos\theta d\theta\int d\phi;$$

noticing, that two successive integrations are called a double integral, and so on, according to the number of successive integrations. It may be added, that the preceding transformation is essentially the same as that of Laplace, at p. 6, vol. II., of the "Mécanique Céleste," and that of Lacroix, at p. 209, vol. II., of his "Traité du Calcul Intégral."

By applying the preceding formula to find the contents of a sphere whose radius is R, it is manifest, as before, that the integral with regard to $d\theta$ must be taken through the whole circumference, which reduces it to

$$\int r^2dr\int\cos\theta d\theta\int d\phi = 2\pi\int r^2dr\cos\theta d\theta;$$

whose integral with regard to θ must be taken from

$$\sin\theta = -1 \quad \text{to} \quad \sin\theta = 1,$$

which reduces it to

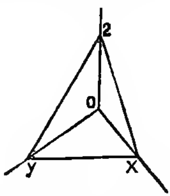
$$2\pi\int r^2 dr\int \cos\theta d\theta = 4\pi\int r^2 dr;$$

whose integral, with reference to r, must be taken from $r = 0$ to $r = \text{R}$, which gives

$$4\pi\int r^2 dr = \frac{4\pi \text{R}^3}{3},$$

for the volume of the sphere.

(17.) We now propose to show how to find the surface of a body or solid, on suppositions like to those in (16), and shall premise the following important proposition:



Thus, let ox, oy, and oz, be three rectangular axes having o for their origin; then, *the square of the numerical value of the face xyz of the triangular pyramid oxyz, equals the sum of the squares of the numerical values of the three remaining faces of the pyramid.*

For representing ox, oy, and oz, severally by a, b, and c, the right triangles oxy, oyz, and oxz, severally give

$$\sqrt{(a^2+b^2)},\ \sqrt{(b^2+c^2)},\ \sqrt{(a^2+c^2)},$$

for the representatives of the sides xy, yz, and xz, of the triangular face xyz of the pyramid.

Hence, since the triangular faces oxy, oyz, and oxz, are severally represented by $\frac{ab}{2}$, $\frac{bc}{2}$, and $\frac{ac}{2}$, we propose to show that the square of the face xyz equals

$$\frac{a^2b^2}{4}+\frac{b^2c^2}{4}+\frac{a^2c^2}{4}=\frac{a^2b^2+b^2c^2+a^2c^2}{4}.$$

If, for brevity, we represent the sides xy, yz, and xz, by A, B, C; by a well-known rule for finding the area of a triangle from its three sides, we shall have the area of the triangle xyz expressed by

$$\left(\frac{A+B+C}{2}\times\frac{B+C-A}{2}\times\frac{A+C-B}{2}\times\frac{A+B-C}{2}\right)^{\frac{1}{2}},$$

whose square equals

$$\frac{(B+C)^2-A^2}{4}\times\frac{A^2-(B-C)^2}{4}$$

$$=\frac{B^2+C^2-A^2+2BC}{4}\times\frac{A^2-(B^2+C^2)+2BC}{4}$$

$$=\frac{2BC+(B^2+C^2-A^2)}{4}\times\frac{2BC-(B^2+C^2-A^2)}{4}$$

$$=\frac{4B^2C^2-(B^2+C^2-A^2)^2}{16}.$$

From the substitutions of the values $\sqrt{(a^2+b^2)}$, $\sqrt{(b^2+c^2)}$, and $\sqrt{(a^2+c^2)}$, of A, B, and C, in the preceding equation, we have the square of the face xyz equal to

$$\frac{4\,(b^2+c^2)\,(a^2+c^2)-4c^4}{16}=\frac{a^2b^2+a^2c^2+b^2c^2}{4};$$

as required. It is clear that the triangles xyo, yzo, and xzo, are severally equal to the projections of the triangle xyz, by perpendiculars upon them. And since, from principles of geometry, the perpendicular from o to the face xyz, multiplied by it, equals the perpendicular oz multiplied by the triangle *yxo*, to which it is perpendicular, each product being three times the pyramid, it follows that the triangle *xyo* equals the triangle *xyz* multiplied by the quotient resulting from the division of the perpendicular from *o* by *oz*, which is clearly the cosine of the inclination of the face *xyz* to the face *xyo*.

Hence, the cosine of the inclination of *xyz* to either of the other faces multiplied by *xyz* equals the other face; consequently, from what has been shown, it follows that the sum of the squares of the cosines of the inclinations of the face *xyz* to each of the other faces equals unity or 1.

Hence, also, any plane in the plane *xyz* is such, that its square equals the sum of the squares of its projections on the three planes *xyo*, *yzo*, and *xzo*.

We will now suppose the curve surface to be touched by a plane at any one of its points, and that an unlimitedly small portion of it at the point of contact, having two of its opposite sides parallel to the plane of x, z, and the other two opposite sides parallel to the plane of y, z, is taken for the differential of the curve surface. Then, the projections of the parallelogram thus formed on the planes x, y, x, z, and y, z, will evidently be parallelograms whose areas may be expressed by the products $dxdy$, $dydz$, and $dxdz$; consequently, from what has been shown, we shall have

$$dx^2dy^2 + dy^2dz^2 + dx^2dz^2 = dx^2dy^2\left\{1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right\}$$

for the square of the differential of the curve surface, and of course if d^2S represents the differential, we shall have

$$d^2S = dxdy\sqrt{\left\{1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right\}}$$

for the required differential of the curve surface.

It may be noticed that $\frac{dz}{dx}$ and $\frac{dz}{dy}$, which suppose z to be a function of x and y, have heretofore been represented, as in Sections 8 and 9, by p and q; agreeably to which, if we please, we may write the preceding equation, according to custom, in the form
$$d^2S = dxdy\sqrt{(1 + p^2 + q^2)}.$$

It may also be noticed, that according to what has been shown, $\sqrt{(1 + p^2 + q^2)} = \sqrt{\left\{1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right\}}$ equals the reciprocal of the cosine of the angle made by the tangent plane with the plane x, y.

To illustrate what has been done, we will apply the formula to find the surface of a sphere whose equation is

$$z^2 + y^2 + x^3 = R^2$$

By taking the partial differential coefficients, we get

$$\frac{dz}{dx} = -\frac{x}{z} \quad \text{and} \quad \frac{dz}{dy} = -\frac{y}{z},$$

which give

$$1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2 = 1 + \frac{x^2}{z^2} + \frac{y^2}{z^2} = \frac{z^2 + y^2 + x^2}{z^2} = \frac{R^2}{z^2};$$

consequently, we shall have

$$d^2S = \frac{Rdxdy}{z} = \frac{Rdxdy}{\sqrt{(R^2 - y^2 - x^2)}} \quad \text{or} \quad d\left(\frac{dS}{dy}\right) = \frac{Rdx}{\sqrt{(R'^2 - x^3)}},$$

by putting $R'^2 = R^2 - y^2$. By taking the integral relatively to x, or by regarding R' as being constant, we have

$$\frac{dS}{dy} = R\int \frac{dx}{\sqrt{(R'^2 - x^2)}} = R \sin^{-1} \frac{x}{R'} = R \sin^{-1} \frac{x}{\sqrt{(R^2 - y^2)}},$$

which, taken from $x = 0$ to $x = \sqrt{(R^2 - y^2)}$, gives $\frac{dS}{dy} = \frac{R\pi}{2}$;

consequently, $dS = \frac{R\pi}{2} dy$, whose integral is $S = \frac{R\pi}{2} y$, which, taken from $y = 0$ to $y = R$, is $\frac{R^2\pi}{2}$, the eighth part of the surface of the whole sphere, which, of course, equals $4R^2\pi$, four times the area of a great circle of the sphere.

Otherwise.—By putting the equation of the spheric surface in the form

$$R^2 = x^2 + y^2 + z^2 = r^2 + z^2,$$

we shall, by the notation at p. 298, get

$$dxdy = rdrd\phi = -zdzd\phi,$$

and thence $d^2S = -zdzd\phi \times \frac{R}{z} = -dzRd\phi$;

whose integral relatively to ϕ must clearly be taken throughout the entire circumference, and gives $ds = -2R\pi dz$; and the integral of this must evidently be taken from $z = -R$ to $z = R$, which gives $4\pi R^2$ for the whole surface of the sphere, the same result as by the preceding method.

(18.) We will now proceed to show the use of arbitrary constants in the development of functions, and in the integration of differential equations, more than has yet been done.

1. To show the use of constants in the development of functions, we will give the following investigation of Taylor's Theorem.

Thus, suppose the differential of any function of $x + h$ may be represented by the form

$$dF(x+h) = F'(x+h)\,dh,$$

when the differential is taken on the supposition that h alone is variable. By taking the integrals of the members of the equation, we have

$$F(x+h) = C + \int F'(x+h)\,dh;$$

in which $F(x+h)$ is the integral of the exact differential $dF(x+h)$, and C the arbitrary constant, while $\int F'(x+h)\,dh$ indicates that the integral of $F'(x+h)\,dh$ is to be found, on the supposition that h alone is regarded as variable. If we determine C on the hypothesis that the integral $\int F'(x+h)\,dh$ vanishes when h equals naught, since $h = 0$ reduces $F(x+h)$ to $F(x)$, and $\int F'(x+h)\,dh$ to naught, we shall have $F(x) = C$. Hence, by substituting this value of C, the equation

$$F(x+h) = C + \int F'(x+h)\,dh$$

is reduced to $F(x+h) = F(x) + \int F'(x+h)\,dh$;

noticing, that $F(x)$ is not supposed to be unlimitedly great. Because $F'(x+h)$ is a function of $x+h$, it follows, from what has been done, that for $F'(x+h)$ we may put

$$F'(x) + \int F''(x+h)\,dh,$$

which reduces the preceding equation to

$$F(x+h) = F(x) + \int F'(x)\,dh + \int dh \int F''(x+h)\,dh$$
$$= F(x) + F'(x)\,h + \int^2 F''(x+h)\,dh^2;$$

since $\int F'(x)\,dh$ becomes $F'(x)\,h$ on account of the constancy of $F'(x)$, and by using $\int^2$ (according to custom) for $\int\int$. Similarly, because $F''(x+h)$ may be represented by $F''(x) + \int F'''(x+h)\,dh$, we have

$$\int^2 F''(x+h)\,dh^2 = \int dh \int F''(x)\,dh + \int^3 F'''(x+h)\,dh^3,$$

and hence

$$F(x+h) = F(x) + F'(x)\frac{h}{1} + F''(x)\frac{h^2}{1.2} + \int^3 F'''(x+h)\,dh^3.$$

If n represents any positive integer, it is manifest that we shall in this way get

$$F(x+h) = F(x) + F'(x)\frac{h}{1} + F''(x)\frac{h^2}{1.2} + F'''(x)\frac{h^3}{1.2.3} + \ldots$$

$$+ F^{n-1}(x)\frac{h^{n-1}}{1.2\ldots.(n-1)} + \int^n F^n(x+h)\,dh^n.$$

To find the values of $F'(x)$, $F''(x)$, $F'''(x)$, &c., we resume the proposed equation

$$dF(x+h) = F'(x+h)\,dh,$$

which gives $$\frac{dF(x+h)}{dh} = F'(x+h),$$

for which we may evidently put

$$\frac{dF(x+h)}{dx} = F'(x+h) = F'(x) + \int F''(x+h)\,dh;$$

for, since x and h enter the function $F(x+h)$ in the same manner, it is clear that the differential coefficient taken by regarding h alone as variable, must be equal to its differential coefficient, taken by regarding x alone as variable.

Because $F'(x)$ enters the preceding equation, like the arbitrary constant C in the equation

$$F(x+h) = C + \int F'(x+h)\,dh,$$

it is manifest that we may determine $F'(x)$ from the equation

$$\frac{dF(x+h)}{dx} = F'(x) + \int F''(x+h)\,dh,$$

on the supposition that when $h = 0$, we must also have

$$\int F''(x+h)\,dh = 0;$$

consequently, by putting $h = 0$, we get

$$F'(x) = \frac{dF(x)}{dx}.$$

Because the equation

$$F'(x+h) = F'(x) + \int F''(x+h)\,dh$$

may be supposed to have been obtained from

$$dF(x+h) = F''(x+h)\,dh,$$

in the same way that

$$F(x+h) = F(x) + \int F'(x+h)\,dh$$

has been derived from

$$dF(x+h) = F'(x+h)\,dh,$$

it is clear that we shall (as before) get

$$F''(x) = \frac{dF'(x)}{dx}.$$

Because $F'(x) = \frac{dF(x)}{dx}$, if dx is constant, it is clear that for $F''(x) = \frac{dF'(x)}{dx}$ we may write $\frac{d^2F(x)}{dx^2}$; in which $\frac{d^2F(x)}{dx^2}$ is called the second differential coefficient of $F(x)$. It is evident that we shall in like manner get

$$F'''(x) = \frac{d^3F(x)}{dx^3}, \quad F^{\text{IV}}(x) = \frac{d^4F(x)}{dx^4},$$

and so on, for the third, fourth, &c., differential coefficients. Hence, we shall have

$$F(x+h) = F(x) + \frac{dF(x)}{dx}\frac{h}{1} + \frac{d^2F(x)}{dx^2}\frac{h^2}{1.2} + \frac{d^3F(x)}{dx^3}\frac{h^3}{1.2.3} +,$$

&c., as in Taylor's Theorem, as required.

It will be perceived that, in the preceding investigation, we have virtually introduced an unlimitedly great number of constants; since there must (essentially) be as many as there are equations like

$$F(x+h) = C + \int F'(x+h)\,dh = F(x) + \int F'(x+h)\,dh,$$

$$F'(x+h) = F'(x) + \int F''(x+h)\,dh, \text{ and so on.}$$

But since these constants all result from $C = F(x)$, or are dependent on C, it is clear that the integral of

$$dF(x+h) = F'(x+h)\,dh$$

contains only one arbitrary constant. Indeed, it is manifest that in

$$\phi(x+h) = \phi(x) + \frac{d\phi(x)}{dx}\frac{h}{1} + \frac{d^2\phi(x)}{dx^2}\frac{h^2}{1.2} +, \text{ \&c.}$$

$$\phi(x), \quad \frac{d(\phi x)}{dx}, \quad \frac{d^2\phi(x)}{dx^2}, \text{ \&c.,}$$

enter as constants; whose values result from $\phi(x)$, or depend on x and the form of the function represented by ϕ.

It is hence evident, that in integrating any differential equation there will be as many constants introduced as there are integrations, which will be arbitrary when they are independent of each other.

2. Supposing an equation between variables and constants to be freed from fractions and radicals, and that its terms are all brought into the first member of the equation, and ordered according to the ascending or descending powers of one or more of the unknown letters, then, if the equation has a term called *the absolute term*, which does not contain any variable, by taking the differential of the equation, the absolute term will disappear from the differential equation; and the pro-

posed equation, sometimes called the *primitive*, is said to have lost a constant in the differential equation, sometimes called the *first derivative* of the proposed equation, by a direct differentiation of the primitive; but if the form of the primitive is changed, so as to make the constant coefficient of any other term of the equation the absolute term of the changed equation, its absolute term will, as before, disappear from its differential equation, which may be called an *indirect derivative* of the proposed equation, which may be said to have resulted from *an indirect differentiation* of the proposed equation. It is hence easy to perceive that there may be as many direct and indirect differential equations obtained from the given primitive, to free it from each of its constants separately, as it contains constants.

Thus, if $y + ax + b = 0$ represents the given equation, having b for its absolute term, then, by a direct differentiation of the equation, we get $dy + adx = 0$ or $\frac{dy}{dx} + a = 0$ for the direct derivative of the proposed primitive, which does not contain the absolute term b. By putting the proposed equation under the form $\frac{y+b}{x} - a = 0$, we have a for its absolute term; then, taking the differentials of the members of this, we have

$$d\frac{y+b}{x} = \frac{d(y+b) \times x - dx(y+b)}{x^2} = 0,$$

or

$$xdy - ydx - bdx = 0,$$

or its equivalent

$$y - \frac{xdy}{dx} + b = 0,$$

which is the indirect derivative of the proposed equation, which is clearly the same result that the elimination of a from

$$y + ax + b = 0 \quad \text{by} \quad \frac{dy}{dx} + a = 0$$

will give; it is also clear that the elimination of $\frac{dy}{dx}$ from the differential equations

$$y - \frac{xdy}{dx} + b = 0 \quad \text{and} \quad \frac{dy}{dx} + a = 0$$

will reproduce the proposed primitive. It is also manifest that the derivative equations

$$\frac{dy}{dx} + a = 0 \quad \text{and} \quad y - \frac{xdy}{dx} + b = 0$$

are entirely distinct from each other; the equivalent of the first

$$dy + adx = 0$$

being immediately integrable, while the integral of the equivalent of the second

$$ydx - xyd + bdx = 0 \quad \left(\text{or} \quad \frac{xdy - ydx}{x^2} - \frac{bdx}{x^2} = 0\right)$$

becomes integrable after it is multiplied by $-\frac{1}{x^2}$, *the factor which is said to be requisite to the integrability* of the indirect derivative, $ydx - xdy + bdx = 0$, of the proposed primitive.

If we take the equation $y + bx + cx^2 = 0$, it is evident that a constant can not be eliminated from it by a single direct differentiation, while the constants b and c can be eliminated by indirect differentiations. For, by putting the equation under the forms

$$\frac{y}{x^2} + \frac{b}{x} + C = 0 \quad \text{and} \quad \frac{y}{x} + cx + b = 0,$$

and taking the differentials, we have

$$d\,\frac{y}{x^2} + d\,\frac{b}{x} = 0 \quad \text{or} \quad \frac{xdy}{dx} - (2y + bx) = 0,$$

and
$$d\,\frac{y}{x} + cdx = 0 \quad \text{or} \quad x\,\frac{dy}{dx} - (y - cx^2) = 0.$$

It is evident that by eliminating $\frac{dy}{dx}$ from these equations, we shall get the primitive equation $y + bx + cx^2 = 0$, which can not be found from the immediate integration of either of the derived equations.

If, for another example, we take the equation

$$y - ax + a^2 = 0;$$

then, by differentiation, we have $dy - adx = 0$ or $\frac{dy}{dx} = a$. Substituting $\frac{dy}{dx}$ for a, in the proposed equation, it becomes

$$y - \frac{xdy}{dx} + \frac{dy^2}{dx^2} = 0,$$

which is of the second degree in $\frac{dy}{dx}$, and of the first order of differentials. Thus we perceive how differential coefficients of the higher orders may sometimes be introduced into differential equations, by eliminating the different powers of a constant from it, by means of the powers of a differential coefficient; but it is manifest from the methods of finding multiple points in Section VII., that they may sometimes be introduced by differentiating as in finding multiple points. (See the examples at p. 191, &c.)

3. We now propose to show how to reduce such integrals as are of the forms $\int^m Xdx^m$, $\int^n Xdx^n$, &c., m and n being positive integers, to simple integrals, expressed by the sign $\int$. Thus,

$$\int^2 \mathrm{X}dx^2 = \int dx \int \mathrm{X}dx = \int dx \int (\mathrm{X}dx + x\mathrm{X}dx - \mathrm{X}xdx)$$
$$= x\int \mathrm{X}dx - \int \mathrm{X}xdx\,;$$

which clearly results from integrating by parts (see p. 260). Similarly,

$$\int^3 \mathrm{X}dx^3 = \int dx \int^2 \mathrm{X}dx^2 = \int (xdx\int \mathrm{X}dx - dx\int \mathrm{X}xdx)$$
$$= \frac{1}{1.2}\,(x^2\int \mathrm{X}dx - 2x\int \mathrm{X}xdx + \int \mathrm{X}x^2dx),$$

$$\int^4 \mathrm{X}dx^4 = \int dx \int^3 \mathrm{X}dx^3$$
$$= \frac{1}{1.2}\int (x^2\int \mathrm{X}dx - 2xdx\int \mathrm{X}xdx + dx\int \mathrm{X}x^2dx)$$
$$= \frac{1}{1.2.3}\,(x^3\int \mathrm{X}dx - 3x^2\int \mathrm{X}xdx + 3x\int \mathrm{X}x^2dx - \int \mathrm{X}x^3dx),$$

and so on, to

$$\int^n \mathrm{X}dx^n = \frac{1}{1.2.3\ldots(n-1)}\,[x^{n-1}\int \mathrm{X}dx - \frac{n-1}{1}\,x^{n-2}\int \mathrm{X}xdx$$
$$+ \frac{(n-1)\,(n-2)}{1\,.\,2}\,x^{n-3}\int \mathrm{X}x^2dx -, \text{ \&c.}],$$

whose law of continuation is manifest. (See Lacroix, vol. II., p. 152.) If for

$$\int \mathrm{X}dx, \int \mathrm{X}xdx, \int \mathrm{X}x^2dx, \text{ \&c.},$$

in the preceding formula, we put

$$\int \mathrm{X}dx + \mathrm{C}, \int \mathrm{X}xdx + \mathrm{C}', \int \mathrm{X}x^2dx + \mathrm{C}'', \text{ \&c.},$$

in which C, C′, C″, &c., are the arbitrary constants, they will represent the complete integrals indicated by $\int^n \mathrm{X}dx^n$; because there will be as many arbitrary constants as there are integrations, and they clearly enter the formula, as they ought to do.

If the constants equal naught, it is clear that the preceding formula is equivalent to

$$\int^n X dx^n = \frac{1}{1.2.3\ldots(n-1)}\int X\,(y-x)^{n-1}dx\,;$$

provided y is regarded as independent of x in the integration, and that the integral is taken from the value of x at the commencement of the integral, to the value of x at the end of it; for which last value (of x) we ought to put y, or y must represent it.

REMARKS.—1. The preceding formula enables us to find limits to the integrals indicated by

$$\int^n F^n\,(x+h)\,dh^n = \int^n \frac{d^nF\,(x+h)}{dx^n}\,dh^n,$$

given in the investigation of Taylor's Theorem, at p. 306.

For X may represent $\dfrac{d^nF\,(x+h)}{dx^n}$, and h may be used for x in the preceding formula; consequently, we shall have

$$\int^n X dh^n = \frac{1}{1.2.3\ldots.(n-1)}\int X\,(y-h)^{n-1}dh.$$

If we put $y-h=yz$, or $h=y\,(1-z)$, we shall have $dh=-ydz$, since y is independent of h; consequently, we shall get

$$\int^n X dh^n = -\frac{1}{1.2.3\ldots.(n-1)}\int X y^n z^{n-1}dz,$$

supposing the integral to be taken from $z=1$ or $h=0$ to $z=0$ or $y=h$. If the limits of the integral are interchanged, it is evident that we shall have

$$\int^n X dh^n = \frac{1}{1.2.3\ldots.(n-1)}\int X y^n z^{n-1}dz.$$

If M and m are the greatest and least values of X (regarded as having the same sign and as finite), in the interval from x to $x+h$, then we shall have

$$\int^n \mathrm{X}dh^n = \frac{h^n}{1.2\ldots.(n-1)}\int \mathrm{X}y^n z^{n-1}\,dz,$$

such that $\frac{\mathrm{M}h^n}{1.2.3\ldots.n}$ and $\frac{mh^n}{1.2.3\ldots.n}$ are its greater and less limits; noticing, that these limits are clearly the limits of the errors committed by rejecting

$$\int^n \mathrm{X}dh^n = \frac{1}{1.2.3\ldots.(n-1)}\int \mathrm{X}y^n z^{n-1}dz.$$

(See Lacroix, vol. III., p. 398.)

2. It is easy to find the integrals indicated by $\int^n \mathrm{X}dx^n$, in such a way that they shall be freed from $\int$, the sign of integration. Thus, since

$$\int^2 \mathrm{X}dx^2 = \int dx \int \mathrm{X}dx$$

$$= \int dx\left(\mathrm{X}x - \frac{d\mathrm{X}}{dx}\frac{x^2}{1.2} + \frac{d^2\mathrm{X}}{dx^2}\frac{x^3}{1.2.3} -, \&c.\right)$$

(see Bernouilli's series at p. 261), and by disregarding the arbitrary constants (for the present), we shall, by integrating by parts, get $\int^2 \mathrm{X}dx^2 =$

$$\mathrm{X}\frac{x^2}{1.2} - \frac{d\mathrm{X}}{dx}\frac{2x^3}{1.2.3} + \frac{d^2\mathrm{X}}{dx^2}\frac{3x^4}{1.2.3.4} - \frac{d^3\mathrm{X}}{dx^3}\frac{4x^5}{1.2.3.4.5} +, \&c.$$

From this result, we, in like manner, get

$$\int^3 \mathrm{X}dx^3 = \int dx \int^2 \mathrm{X}dx^2 = \int dx\left(\mathrm{X}\frac{x^2}{1.2} - \frac{d\mathrm{X}}{dx}\frac{2x^2}{1.2.3} +, \&c.\right);$$

which, integrated by parts, as before, gives

$$\int^3 \mathrm{X}dx^3 = \mathrm{X}\frac{x^3}{1.2.3} - \frac{d\mathrm{X}}{dx}\frac{3x^4}{1.2.3.4} + \frac{d^2\mathrm{X}}{dx^2}\frac{6x^5}{1.2.3.4.5} -, \&c.$$

Proceeding in this way, and supplying the arbitrary constants, it is easy to perceive that we shall have

$$\int^n X dx^n = X \frac{x^n}{1.2\ldots.n} - \frac{dX}{dx}\frac{nx^{n+1}}{1.2\ldots.(n+1)}$$

$$+ \frac{d^2X}{dx^2}\frac{\frac{n(n+1)}{1.2}x^{n+2}}{1.2\ldots.(n+2)} - \frac{d^3X}{dx^3}\frac{\frac{n(n+1)(n+2)}{1.2.3}x^{n+3}}{1.2\ldots.(n+3)} + \&c.$$

$$+ Cx^{n-1} + C'x^{n-2} + C''x^{n-3} + \ldots. + C^{n-1}x^0,$$

C, C′, &c., being the arbitrary constants. (See Lacroix, vol. II., pp. 154 and 155.)

Being now prepared, we will give a short section on the Calculus of Variations.

SECTION II.

FIRST PRINCIPLES OF THE CALCULUS OF VARIATIONS.

(1.) If V is an arbitrary variable, which depends on a constant; then, if in consequence of a change in the constant it becomes V′, the difference $V' - V$, represented by δV, is called the variation of V, which is expressed by writing δ, called *the characteristic of variations*, before or to the left of V. If $\phi(V)$ represents any function of V, and the algebraic sum of all the changes in the value of $\phi(V)$ that result from the separate variation $V' - V$, represented by δV, of each V in $\phi(V)$ is taken, it will represent what is called the variation of $\phi(V)$; which, as before, is expressed by writing the characteristic δ before or to the left of the function; so that $\delta\phi(V)$ stands for the variation of the function $\phi(V)$.

(2.) From a comparison of the preceding definitions with those of a differential of a variable and a function of it [see (4) at p. 2], it is easy to perceive that we shall have

$$\delta\phi(V) = \frac{d\phi(V)}{dV}\,\delta V;$$

$\frac{d\phi(V)}{dV}$ being the differential coefficient, regarding V as being the independent variable. Hence we shall have

$$\frac{\delta\phi(V)}{\delta V} = \frac{d\phi(V)}{dV};$$

which shows that *the variational and differential coefficients* *of a function, with reference to the same variable, are equal to each other.*

(3.) Since from (1.) $V' = V + \delta V$, we have, from Taylor's Theorem,

$$\phi(V') = \phi(V + \delta V) = \phi(V) + \frac{d\phi(V)}{dV}\delta V +, \&c.;$$

which, by retaining only the term that contains the simple power of δV, becomes

$$\phi(V') = \phi(V) + \frac{d\phi(V)}{dV}\delta V,$$

which clearly shows that $\phi(V')$ must be of a different form from $\phi(V)$, since δV results from the change of a constant contained in V.

Hence, if we represent the proper form of the first member of the equation by $\psi(V')$, we shall get

$$\psi(V') = \phi(V) + \frac{d\phi V}{dV}\delta V;$$

which gives $$\psi(V') - \phi(V) = \frac{d\phi(V)}{dV}\delta V.$$

Since $\frac{d\phi(V)}{dV}\delta V$ is, according to what has been shown, equal to $\delta\,\phi(V)$, we shall hence get

$$\psi(V') - \phi(V) = \delta\phi(V).$$

By taking the differentials of the members of this equation, we have $$d\psi(V') - d\phi(V) = d\delta\phi(V);$$

or since $d\psi(V')$ is a change of the form $d\phi(V)$, we shall have $$d\psi(V') - d\phi(V) = \delta d\phi(V),$$

and thence $$d\delta\phi(V) = \delta d\phi(V);$$

and with equal facility we get

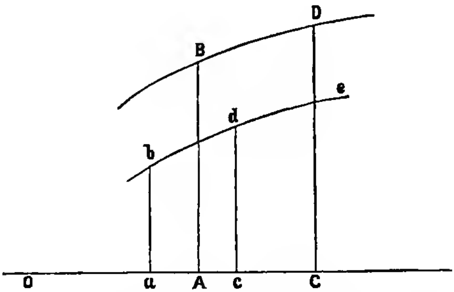
$$d^n\delta\phi(V) = \delta d^n\phi(V),$$

n being a positive integer. Hence, in

$$d^n\delta\phi(V) = \delta d^n\phi(V),$$

or in any expression to which d^n and δ are prefixed, we may clearly interchange d and δ, the characteristics of differentials and variations, without affecting the value of the result; noticing, that this is usually considered as being the fundamental principle of the Calculus of Variations.

On account of the importance of what has been done, in what is to follow, we propose to illustrate it geometrically.



Thus, if the line OC is taken for the line of the abscissas, on which the positive values of V are estimated from the origin O, toward the right; then ab, being drawn as an ordinate to the curve be, representing the value of $\phi(V)$, which corresponds to $Oa = V$, by changing Oa or V into OA or V', and drawing AB parallel to ab to represent the changed value of $ab = \phi(V)$ as an ordinate $\psi(V')$, in the changed curve BD, we shall have $\psi(V') = \phi(V)$, the variation of ab represented by $AB - ab$ in the figure.

Similarly, Oc and OC representing other values of V and V', we shall have $CD - cd$ for the representative of the corresponding value of $\psi(V') - \phi(V)$, which may be regarded as consecutive to the preceding value.

Hence, we shall have

$$(\mathrm{CD}-cd)-(\mathrm{AB}-ab)=(\mathrm{CD}-\mathrm{AB})-(cd-ab);$$

the first member of this equation, from the definitions at page 2, being the differential of

$$(\mathrm{AB}-ab)=\psi\mathrm{V}'-\phi\mathrm{V}=\delta\phi(\mathrm{V}),$$

since $(\mathrm{AB}-ab)$ is on the same curves with its consecutive value $(\mathrm{CD}-cd)$; while $(cd-ab)$ in the second member of the equation has $(\mathrm{CD}-\mathrm{AB})$ for its consecutive value, which is taken in the curve BD and not in the curve be; and of course, since $(cd-ab)=d\phi(\mathrm{V})$, we shall have

$$(\mathrm{CD}-\mathrm{AB})-(cd-ab)$$

expressed by $\delta d\phi(\mathrm{V})$. Hence, from what has been done, we shall have $d\delta\phi(\mathrm{V})=\delta d\phi(\mathrm{V})$; which agrees with what has been shown, from other considerations.

Again, since $\mathrm{O}a=\mathrm{V}$, and $ac=d\mathrm{V}$, and $c\mathrm{C}=\delta(\mathrm{V}+d\mathrm{V})$,

we shall have $\quad \mathrm{OC}=\mathrm{V}+d\mathrm{V}+\delta(\mathrm{V}+d\mathrm{V})$;

also, from $\quad \mathrm{O}a=\mathrm{V}$, and $a\mathrm{A}=\delta\mathrm{V}$,

together with $\quad \mathrm{AC}=d(\mathrm{V}+\delta\mathrm{V})$,

we have $\quad \mathrm{OC}=\mathrm{V}+\delta\mathrm{V}+d(\mathrm{V}+\delta\mathrm{V})$.

Hence, from equating these values of OC, we have

$$\mathrm{V}+d\mathrm{V}+\delta(\mathrm{V}+d\mathrm{V})=\mathrm{V}+\delta\mathrm{V}+d(\mathrm{V}+\delta\mathrm{V}),$$

which is easily reduced to $\delta d\mathrm{V}=d\delta\mathrm{V}$.

If AB and CD coincide, in direction, with ab and cd, or if A falls on a and C on c, it is clear that the equation $\delta d\mathrm{V}=d\delta\mathrm{V}$ will not exist.

(4.) There is an analogous principle, with reference to the signs of integration and variation, which we will now proceed to notice.

Thus, if we put the integral indicated by $\int u$ equal to V, by taking the differential we shall have $u = dV$; whose variation gives

$$\delta u = \delta dV = (\text{by interchanging } \delta \text{ and } d)\ d\delta V,$$

whose integral gives

$$\int \delta u = \delta V = \delta \int u.$$

In like manner, by representing the nth integral $\int^n u$ by V, and taking the nth differentials of these equals, we shall have $u = d^n V$; whose variation is

$$\delta u = \delta d^n \delta V = d^n \delta V,$$

whose nth integral gives

$$\int^n \delta u = \delta V = \delta \int^n u.$$

Hence, if the characters $\int^n$ and δ are prefixed to any expression, *they may be interchanged without affecting its value.*

(5.) If t in any calculation represents the independent variable, then, since $d\delta t = \delta dt$, and that dt is constant, we shall have $\delta dt = 0$, since δt is invariable; consequently, from

$$d\delta t = \delta dt = 0 \quad \text{we have} \quad d\delta t = 0,$$

whose integral is $\delta t = \text{const.}$

Hence, *the variation of the independent variable is constant, or invariable.*

(6.) To illustrate what has been done, and to show the nature of variations more fully, we will take the following

EXAMPLES.

1. To find the variations of $y^{\frac{m}{n}}$, $x^{-\frac{p}{q}}$, xy, and $\frac{y}{x}$.

By differentiating and using δ for d (see p. 316), we readily get

$$\frac{m}{n}y^{\frac{m}{n}-1}\delta y, \ -\frac{p}{q}x^{-\frac{p}{q}-1}\delta x, \ y\delta x + x\delta y, \ \text{and} \ \frac{x\delta y - y\delta x}{x^2},$$

for the variations.

2. Given $t = \dfrac{ydx}{dy}$ and $s = \dfrac{ydy}{dx}$, to find the variations of t and s, when dy is regarded as constant.

Differentiating, and using δ for d, we get

$$\delta t = \frac{\delta y dx + y\delta dx}{dy} = \frac{\delta y dx + yd\delta x}{dy},$$

and
$$\delta s = \frac{\delta y dy dx - y dy d\delta x}{dx^2}.$$

3. Given $du = mdx + m'd^2x + ndy + n'd^2y + pd^3z$, to find the variation of u.

This is clearly effected by changing either d in the several terms into δ, which gives

$$\delta u = m\delta x + m'd\delta x + n\delta y + n'\delta dy + pd\delta dz =$$

(agreeably to what has been done)

$$m\delta x + m'd\delta x + n\delta y + n'd\delta y + pd^2\delta z,$$

as required. This is the same as to change the last d into δ.

4. To free the variations under the sign $\int$, in the integrals

$$\int p\delta dx = \int pd\delta x, \quad \text{and} \quad \int q\delta d^2y = \int qd^2\delta y,$$

from the sign d, of differentiation.

Integrating by parts, these expressions are readily reduced to

$$\int pd\delta x = p\delta x - \int dp\delta x,$$

and $\displaystyle\int qd^2\delta y = qd\delta y - \int dqd\delta y = qd\delta y - dq\delta y + \int d^2q\delta y,$

as required.

5. To find the variation of the integral $\int\sqrt{dx^2+dy^2}$, which indicates the length, or rectification, of an indefinite arc of a plane curve, or line, when referred to rectangular co-ordinates; noticing, that such an integral is sometimes called *an indefinite or unlimited integral.*

By taking the variation, regarding both dx and dy as variable, and interchanging the signs of integration and variation, we have

$$\delta\int\sqrt{dx^2+dy^2}=\int\delta\sqrt{dx^2+dy^2}$$

$$=\int\left(\frac{dx}{\sqrt{dx^2+dy^2}}\,\delta dx+\frac{dy}{\sqrt{dx^2+dy^2}}\,\delta dy\right)$$

(by interchanging δ and d, and integrating by parts)

$$=\int\frac{dx}{\sqrt{dx^2+dy^2}}\,d\delta x+\int\frac{dy}{\sqrt{dx^2+dy^2}}\,d\delta y$$

$$=\frac{dx}{\sqrt{dx^2+dy^2}}\,\delta x+\frac{dy}{\sqrt{dx^2+dy^2}}\,\delta y-$$

$$\int d\,\frac{dx}{\sqrt{dx^2+dy^2}}\,\delta x-\int d\,\frac{dy}{\sqrt{dx^2+dy^2}}\,\delta y.$$

It will be perceived that the preceding integral consists of two sorts of terms: one of which, that clearly relates to the limits of the integral, is freed from the sign $\int$; while the other terms are under $\int$, or their integrals are to be taken.

If x', y', and x'', y'', are the co-ordinates of the first and last extremities of the integral, then, the integral taken between the preceding limits, becomes

$$\delta \int_{x', y'}^{x'', y''} \sqrt{dx^2 + dy^2} = \frac{dx''}{\sqrt{dx''^2 + dy''^2}} \delta x'' + \frac{dy''}{\sqrt{dx''^2 + dy''^2}} \delta y''$$

$$- \left(\frac{dx'}{\sqrt{dx'^2 + dy'^2}} \delta x' + \frac{dy'}{\sqrt{dx'^2 + dy'^2}} \delta y' \right)$$

$$- \int d \frac{dx}{\sqrt{dx^2 + dy^2}} \delta x - \int d \frac{dy}{\sqrt{dx^2 + dy^2}} \delta y.$$

If the extremities of the integral are fixed points; then, it is evident that $\delta x''$, $\delta y''$, $\delta x'$, $\delta y'$, will each equal naught, and the integral will be reduced to

$$\delta \int_{x', y'}^{x'', y''} \sqrt{dx^2 + dy^2} =$$

$$- \int d \frac{dx}{\sqrt{dx^2 + dy^2}} \delta x - \int d \frac{dy}{\sqrt{dx^2 + dy^2}} \delta y.$$

If the extremities of the integral are always on given lines, then $\delta x''$, $\delta y''$, will be connected by the equation of the line at the end of the integral, while $\delta x'$, $\delta y'$, will be connected by the equation of the line at its first extremity.

If the integral is to be exact or freed from the sign $\int$, so as to leave δx and δy arbitrary, then, it is clear that we must have the separate equations

$$\int d \frac{dx}{\sqrt{dx^2 + dy^2}} \delta x = 0 \quad \text{and} \quad \int d \frac{dy}{\sqrt{dx^2 + dy^2}} \delta y = 0,$$

and because δx and δy must be arbitrary, these equations must be satisfied by assuming

$$d \frac{dx}{\sqrt{dx^2 + dy^2}} = 0 \quad \text{and} \quad d \frac{dy}{\sqrt{dx^2 + dy^2}} = 0,$$

whose integrals give

$$\frac{dx}{\sqrt{dx^2 + dy^2}} = \text{const.} \quad \text{and} \quad \frac{dy}{\sqrt{dx^2 + dy^2}} = \text{const.};$$

consequently, eliminating $\sqrt{dx^2+dy^2}$ from these equations, we have $\frac{dy}{dx}=a=\text{const.}$, or $dy=adx$,

whose integral gives $y=ax+b$, the equation of a straight line, whose constants are a and b. Hence, the variation of the proposed integral being exact and its limits fixed points, it is evidently reduced to naught, or

$$\delta\int_{x',\,y'}^{x'',\,y''}\sqrt{dx^2+dy^2}=0.$$

It is also manifest, that the straight line represented by

$$y=ax+b, \quad \text{makes} \quad \int_{x',\,y'}^{x'',\,y''}\sqrt{dx^2+dy^2},$$

a minimum; such, that its value can easily be found from the co-ordinates x', y', and x'', y'', of the fixed points at the extremities of the integral. For by putting x' and y' in $y=ax+b$ we have $y'=ax'+b$ and by putting x'' and y'' for x and y in $y=ax+b$ we also have $y''=ax''+b$; consequently, the solution of the equations

$$y'=ax'+b \quad \text{and} \quad y''=ax''+b,$$

will give the values of a and b, and thence the required straight line can be drawn.

Still supposing $\delta\int_{x',\,y'}^{x'',\,y''}\sqrt{dx^2+dy^2}$

to be exact, and to be put equal to naught, we shall, as before, have $y=ax+b$, the equation of the straight line, together with the equation

$$\frac{dx''}{\sqrt{dx''^2+dy''^2}}\,\delta x''+\frac{dy''}{\sqrt{dx''^2+dy''^2}}\,\delta y''=0,$$

$$-\left(\frac{dx'}{\sqrt{dx'^2+dy'^2}}\,\delta x'+\frac{dy'}{\sqrt{dx'^2+dy'^2}}\,\delta y'\right)=0;$$

which results from the variations at the limits of the proposed integral.

If the limiting curves at the extremities of the integral are independent of each other, then it is clear that the preceding equation reduces to the two separate equations

$$dx''\delta x'' + dy''\delta y'' = 0 \quad \text{and} \quad dx'\delta x' + dy'\delta y' = 0,$$

or their equivalents

$$\frac{dy''}{dx''}\frac{\delta y''}{\delta x''} + 1 = 0 \quad \text{and} \quad \frac{dy'}{dx'}\frac{\delta y'}{\delta x'} + 1 = 0;$$

noticing, that $\delta x''$, $\delta y''$, and $\delta x'$, $\delta y'$, are supposed to belong to the limiting lines, while dx'', dy'', and dx', dy', belong to the extremities of the straight line $y = ax + b$. Hence, from the equation of the straight line we have $\frac{dy'}{dx'} = a$, where it meets the first limiting line, which reduces

$$\frac{dy'}{dx'}\frac{\delta y'}{\delta x'} + 1 = 0 \quad \text{to} \quad \frac{\delta y'}{\delta x'} = -\frac{1}{a},$$

which shows that the straight line, from well-known principles, must cut the first limiting line perpendicularly; and, in like manner, from

$$y = ax + b \quad \text{and} \quad \frac{dy''}{dx''}\frac{\delta y''}{\delta x''} + 1 = 0,$$

it may be shown that the right line must also cut the second limiting line perpendicularly.

Supposing the co-ordinates of either extremity of the integral, as x' and y' are given; then since $\delta x' = 0$ and $\delta y' = 0$, the preceding equations will be reduced to

$$y = ax + b \quad \text{and} \quad \frac{dy''}{dx''}\frac{\delta y''}{\delta x''} + 1 = 0;$$

consequently, the straight line must be drawn from the point whose co-ordinates are x' and y', perpendicular to the second limiting line. Hence, when the limiting lines are straight and in the same plane, they must be parallel; for, otherwise, the minimum integral will evidently be impossible.

REMARKS.—1. We have dwelt at some length on this example, because of its simplicity and great use in showing how to find the variations of indefinite integrals, preparatory to the determination of the forms of those that, between given limits, admit of maxima or minima values.

2. If the limiting lines are not independent of each other, then the equation, which expresses their dependence, must be noticed; and thence the solution may be obtained.

3. If our object had been merely to find the nature of the integral in 5, it is easy to perceive that $\int\sqrt{dx^2 + dy^2}$ might have been written in the form

$$\int\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \times dx,$$

and the variation taken by regarding y as a function of x, supposing dx to be constant, or x to be the independent variable, which would have led to $y = ax + b$, the equation of a straight line, as at p. 324; but this process would not have indicated the manner in which the line must cut its limiting lines, as by the preceding method.

(7.) We will now proceed, according to what is sometimes regarded as the particular object of the method of variations, to consider the maxima and minima of Indefinite Integrals.

1. Let $\int V = u$ represent the integral of any differential;

such, that the form of $\int V$ in terms of its variables is to be found, so as to satisfy certain maxima or minima conditions; then, since it is supposed that the integral can not be expressed except by the sign $\int$, it may be regarded as being of an indefinite form.

2. If u receives a small variation or change of form, in consequence of small changes in the relations of its variables; then, from the generalization of Taylor's Theorem, as at pp. 21 to 23, by using δ for d, it is clear that u will become

$$u + \delta u + \frac{\delta^2 u}{1.2} + \frac{\delta^3 u}{1.2.3} +, \text{ \&c.};$$

such, that δu contains the simple powers of the variations of the variables in u, $\delta^2 u$ contains two dimensions of the same variations, $\delta^3 u$ contains three of their dimensions, and so on. (See Lacroix, vol. II., p. 788.)

Hence (see p. 94, &c.), by a process of reasoning analogous to that used when treating of the maxima and minima of definite forms, it evidently follows that, in order to find the maxima and minima, we must assume

$$\delta u = \delta \int V = \int \delta V \text{ (between proper limits) } = 0;$$

that is, the coefficient of each arbitrary variation under the sign $\int$ must be put equal to naught; noticing, that the forms of u, which make $\delta^2 u$ negative, correspond to maxima, and those which make it positive, give minima.

3. By calculating $\int \delta V$ the integral of the variation of V, it will be found (as in Ex. 5, at p. 322) to consist of two parts; one of which, containing in its most general form (or

when the variables in u or $\int V$ are regarded as being independent of each other) as many terms as there are independent variables under the sign $\int$, each of these terms having the variation of one of the independent variables for a factor, while the remaining part is freed from $\int$, and results from taking the integral of the variation of the proposed integral from its first to its second limit. Because the terms under the sign $\int$ are clearly independent of those without it, and of each other, it is manifest that to reduce $\int \delta V$ to naught, we must put the coefficient of each variation under $\int$ equal to naught. Hence we shall have as many equations as $\int V$ or u is conceived to contain independent variables, which, by the required integrations, as in 5, will be reduced to one less in number than before, or than $\int V$ has been supposed to contain independent variables, which is evident from the consideration that the form of $\int V$ will be determined.

As to the terms of $\int \delta V$, that are freed from $\int$, they must be reduced to naught, and treated in ways very analogous to those used in 5, at p. 322, &c.

Hence, the manner of satisfying $\int \delta V = 0$, becomes too evident to require any further explanation.

EXAMPLES.

1. Supposing a heavy body to descend from one point to another, not in the same vertical line, it is proposed to find

the nature of the line in which the body may descend in the least time.

Let x and y represent the rectangular co-ordinates of the place of the body, at any time t of its motion from the commencement; then, if $\sqrt{dx^2 + dy^2} = ds =$ the differential of the described arc of the sought curve, and v the velocity acquired by the body in its descent, and dt the differential of the time, we shall, from well-known principles of mechanics, have $dt = \frac{ds}{v}$; or, by taking the integral from the time of the body's leaving the highest point to its arrival at the lowest, we shall have $t = \int \frac{\sqrt{dx^2 + dy^2}}{v}$, which is to be a minimum.

If $g = 32\frac{1}{6}$ feet, and y the vertical descent in the time t; then (from p. 154 of Young's "Mechanics," or any of the common works on Mechanics), we shall have $v = \sqrt{2gy}$; and shall thence get

$$t = \int \frac{\sqrt{dx^2 + dy^2}}{\sqrt{2gy}},$$

which must be a minimum; or, since $2g$ is constant, it is evident that when t is a minimum,

$$t\sqrt{2g} = \int \frac{\sqrt{dx^2 + dy^2}}{\sqrt{y}},$$

is also a minimum.

It is hence manifest that for $\int V$ we must here take $\int \sqrt{\frac{dx^2 + dy^2}{y}}$; and that x and y may be regarded as being independent of each other.

Since (from what has been shown) we have

$$\delta\int\sqrt{\frac{dx^2+dy^2}{y}}=\int\delta\sqrt{\frac{dx^2+dy^2}{y}},$$

we shall get, by taking the variations, $\sqrt{2g}\,\delta t = 0$, because t is to be a minimum, or its variation equal to naught; which gives

$$\delta\int\sqrt{\frac{dx^2+dy^2}{y}}=\int\left(\frac{dx}{ds\sqrt{y}}\,\delta dx+\frac{dy}{ds\sqrt{y}}\,\delta dy-\frac{1}{2}\frac{ds}{\sqrt{y^3}}\,\delta y\right)$$

$$=\int\frac{dx}{ds\sqrt{y}}\,d\delta x+\int\left(\frac{dy}{ds\sqrt{y}}\,d\delta y-\frac{1}{2}\frac{ds}{\sqrt{y^3}}\,\delta y\right)=0.$$

Hence, integrating this equation by parts, and using C for the arbitrary constant, we shall have

$$\frac{dx}{ds\sqrt{y}}\,\delta x+\frac{dy}{ds\sqrt{y}}\,\delta y-\int d\,\frac{dx}{ds\sqrt{y}}\,\delta x$$

$$-\int\left(d\,\frac{dy}{ds\sqrt{y}}+\frac{1}{2}\frac{ds}{\sqrt{y^3}}\right)\delta y+\text{C}=0.$$

If x', y', and x'', y'', represent the co-ordinates of the body at its highest and lowest points, we shall, from the elimination of C from the equation, have

$$\frac{dx''}{ds''\delta y''}\,\delta x''+\frac{dy''}{ds''\sqrt{y''}}\,\delta y''-\left(\frac{dx'}{ds'\sqrt{y'}}\,\delta x'+\frac{dy'}{ds'\sqrt{y'}}\,\delta y'\right)$$

$$-\int d\,\frac{dx}{ds\sqrt{y}}\,\delta x-\int\left(d\,\frac{dy}{ds\sqrt{y}}+\frac{1}{2}\frac{ds}{\sqrt{y^3}}\right)\delta y=0.$$

This equation may be satisfied by putting the coefficient of δx under the sign $\int$ equal to naught, or $d\,\dfrac{dx}{ds\sqrt{y}}=0$, whose integral may be represented by $\dfrac{dx}{ds\sqrt{y}}=\dfrac{1}{\sqrt{a}}=$ the arbitrary constant; and because this equation is sufficient to determine the curve described by the body, we may reject

$$\int \left(d \frac{dy}{ds \sqrt{y}} + \frac{1}{2} \frac{ds}{\sqrt{y^3}} \right) \delta y,$$

or assume $\delta y = 0$; and because the first and last points of the curve described by the body in its descent are given, we shall have

$$\delta x' = 0, \quad \delta y' = 0, \quad \delta x'' = 0, \quad \delta y'' = 0;$$

consequently, the above variational equation is fully satisfied.

To determine the curve described from the equation

$$\frac{dx}{ds \sqrt{y}} = \frac{1}{\sqrt{a}},$$

by taking its square, we have

$$\frac{dx^2}{ds^2 y} = \frac{1}{a} \quad \text{or} \quad \frac{dx^2}{ds^2} = \frac{y}{a},$$

which, since $ds^2 = dx^2 + dy^2$, gives $\frac{dy^2}{ds^2} = \frac{a - y}{a}$. Putting $a - y = z$ and $dy = - dz$, this becomes

$$\frac{dz^2}{ds^2} = \frac{z}{a} \quad \text{or} \quad \frac{dz}{ds} = \sqrt{\frac{z}{a}},$$

which reduces to $dz z^{-\frac{1}{2}} = \frac{ds}{\sqrt{a}}$; whose integral gives

$$2\sqrt{az} = s \quad \text{or} \quad s^2 = 4az,$$

which needs no correction, supposing the arc s to be estimated from the lowest of the given points upward, z being the abscissa corresponding to the arc s. It is manifest, from what is done at p. 150, that $s^2 = 4az$ is the equation of a cycloid having its vertex at the lowest of the given points, and a the perpendicular from the lowest point to the axis of x, supposed to be horizontal and to pass through the highest point, for its axis or the diameter of the generating circle; and the point of the axis of x between the highest point and the perpendicular a, is manifestly the semibase of the cycloid.

The same can be shown from

$$\frac{dx}{ds} = \sqrt{\frac{y}{a}} \quad \text{and} \quad \frac{dz}{ds} = \sqrt{\frac{z}{a}};$$

which give

$$\frac{dx}{dz} = \sqrt{\frac{y}{z}} = \sqrt{\frac{a-z}{z}},$$

or

$$dx = \frac{adz - zdz}{\sqrt{az - z^2}} = \frac{\frac{a}{2}dz - zdz}{\sqrt{az - z^2}} + \frac{a}{2}\frac{dz}{\sqrt{az - z^2}}$$

$$= d\sqrt{az - z^2} + \text{arc rad } \frac{a}{2} \text{ versin } z;$$

whose integral gives

$$x = \sqrt{az - z^2} + \text{arc rad } \frac{a}{2} \quad \text{and} \quad \text{versin } z,$$

which agrees with the well-known equation of the cycloid, when the origin of the co-ordinates is at its vertex, and its axis is that of z. (See p. 150.)

REMARKS.—1st. If the body is to move in a vertical plane from a higher to a lower line or curve, in the shortest time, then, when the lines are so placed that the solution is possible, it may be shown, as in example 5, at p. 322, that the cycloid must cut the limiting lines perpendicularly: also, if the body is to move from a point to a lower line, it must move in a cycloid which cuts the line at right angles.

2d. We may find the time of descent from the highest to the lowest of the given points, as follows:

Thus, by taking the differential of $s^2 = 4az$, we get

$$2sds = 4adz \quad \text{or} \quad ds = \frac{2adz}{\sqrt{z}} = dz\sqrt{\frac{a}{z}};$$

consequently, since

$$v = \sqrt{2gy} = \sqrt{2g(a-z)}, \quad \text{or} \quad ds = dz\sqrt{\frac{a}{z}}$$

we shall have

$$dt = -\frac{ds}{\sqrt{2g(a-z)}} = -\sqrt{\frac{a}{2g}}\frac{dz}{\sqrt{(az-z^2)}},$$

by using $-$ in the right member, because z decreases when t increases. It is easy to perceive that this equation is equivalent to the form

$$dt = -d \text{ arc}\left(\text{rad } \frac{a}{2} \text{ versin } z\right) \div \frac{a}{z}.$$

If $\frac{T}{2}$ represents the time of descent of the body from the highest to the lowest point, then, since the arc whose radius $= 1$ and versin $= 2$ is $\pi = 3.14159$, &c., $=$ the semi-circumference, whose rad $= 1$, and that the arc whose versin $= 0$ is also naught, by taking the integral of the preceding differential equation from the arc whose versin $= 0$, we shall get $\frac{T}{2} = \pi\sqrt{\frac{a}{2g}}$, as required.

If z' is supposed to be unlimitedly small, the preceding value of $\quad x' = \sqrt{az' - z'^2} + \text{arc (rad } \frac{a}{2} \text{ versin } z')$,

on account of the comparative smallness of z'^2, and because arc rad $= \frac{a}{2}$ and versin z' differs insensibly from its chord $\sqrt{az'}$, may evidently be reduced to $x' = 2\sqrt{az'}$ very nearly. Now, if the arc of the cycloid corresponding to z' is represented by s', the equation $s^2 = 4az$ becomes $s'^2 = 4az'$, whose square root gives $s' = 2\sqrt{az'}$; consequently, when z' is so small that z'^2 may be regarded as an infinitesimal in comparison to z', we shall have $x' = s'$ very nearly. Hence, because $s'^2 = 4az'$ gives $4a = \frac{s'^2}{z'}$, we shall have $4a = \frac{x'^2}{z'}$; consequently (from a well-known property of the circle), $4a$ equals the diameter of a circle which has the same curvature as the cycloid at its lowest point, or vertex.

Because $s' = 2\sqrt{az'}$ is sensibly the same as a circular arc whose radius is $2a$ and height z', it clearly follows, from what has been shown, that the line of descent down the (infinitesimal) circular arc height z' and rad $= 2a$, differs insensibly from $\frac{T}{2} = \pi\sqrt{\frac{a}{2g}}$.

From this equation, we have

$$T = 2\pi\sqrt{\frac{a}{2g}} = \pi\sqrt{\frac{2a}{g}},$$

which equals the time of an infinitesimal vibration of a pendulum, whose length is $2a$. Hence, if the vertical distance of any two points is denoted by a, it clearly follows from $\frac{T}{2} = \pi\sqrt{\frac{a}{2g}}$, *that the least time in which it is possible for a heavy body to pass from the higher to the lower point, equals the time of one vibration of the pendulum whose length is* $\frac{a}{2}$, *or half the time of one vibration of the pendulum whose length is* $2a$.

3d. The question here treated of, is sometimes called the Problem of the Brachystochrone, or the line of quickest descent.

2. Two points in a vertical plane are connected by a line of uniform diameter and density, to find its nature when its center of gravity is lowest.

Let the line be referred to the axes of x and y in its own plane; the axis of x being horizontal and y vertical, and directed downward.

Then, ds being the differential of the length of the line, $\int ds =$ const. when taken throughout its entire length, will be one of the conditions of the question; and $\int yds =$ max., when taken through the entire length of the line, will be the other condition, since this integral, divided by the length of the line, expresses the descent of the center of gravity.

If we multiply the first condition by the constant a, and add the product to the second condition, the two will clearly, since the first is constant, be reduced to the single condition,

$$a\int ds + \int y ds = \int (a+y)\, ds = \text{max.}$$

By taking the variation, we have

$$\begin{aligned}\delta \int (a+y)\, ds &= \int \delta\,(a+y)\, ds \\ &= \int \left(ds\delta y + (a+y)\frac{dx}{ds}\,\delta dx + (a+y)\frac{dy}{ds}\,\delta dy\right) \\ &= \int (a+y)\frac{dx}{ds}\, d\delta x + \int [ds dy + (a+y)\, d\delta y],\end{aligned}$$

which integrated by parts gives

$$\begin{aligned}&(a+y)\frac{dx}{ds}\,\delta x + (a+y)\frac{dy}{ds}\,dy - \int d\,(a+y)\frac{dx}{ds}\,\delta x \\ &\quad - \int\left(-\,ds + d\,(a+y)\frac{dy}{ds}\right)\delta y + \text{C} = 0,\end{aligned}$$

C being the constant.

Since the extremities of the integral are given points, the part of the integral without the sign $\int$ vanishes, and the constant $= 0$; also, since δx and δy, under the sign $\int$, are arbitrary and independent of each other, we must have

$$d\left\{(a+y)\frac{dx}{ds}\right\} = 0, \quad \text{and} \quad -\,ds + d\left\{(a+y)\frac{dy}{ds}\right\} = 0.$$

The integral of the first of these gives

$$(a+y)\frac{dx}{ds} = b = \text{const.},$$

which gives

$(a + y)^2\, dx^2 = b^2\,(dx^2 + dy^2)$, or $[(a + y)^2 - b^2]\, dx^2 = b^2 dy^2$,

or $$dx = \frac{bdy}{\sqrt{(a + y)^2 - b^2}};$$

and by putting the second equation in the form

$$d\,(a + y)\,\frac{dy}{ds} = ds,$$

and integrating, we have

$$(a + y)\,\frac{dy}{ds} = s + \mathrm{C},$$

C being the arbitrary constant.

If $\frac{dy}{ds} = 0$ when $s = 0$, we have

$$(a + y)\,\frac{dy}{ds} = s,$$

since $\mathrm{C} = 0$; and thence $2ady + 2ydy = 2sds$,

whose integral gives $$2ay + y^2 = s^2,$$

which needs no correction, supposing s and y to commence together and to be reckoned upward; noticing, that the origin of the co-ordinates is clearly at the vertex, since $\frac{dy}{ds} = 0$ at the origin. The preceding equation is the common catenary, the well-known curve, into which a uniform chain of unlimitedly small, short links, when suspended from its extreme points, will form itself; and it is also well known that the equation $$dx = \frac{bdy}{\sqrt{(a + y)^2 - b^2}},$$ previously found, is another form of the equation of the same curve.

Remarks.—Because the length of the curve in this question is given in addition to the maximum condition, the question is said to fall under the class of what are called *isoperimetrical* questions.

3. To find the relation between x and y, when the integral $\int \frac{ydy^3}{dx^2+dy^2}$, taken between proper limits, is a minimum.

Here $$\delta \int \frac{ydy^3}{dx^2+dy^2} = \int \delta \frac{ydy^3}{dx^2+dy^2}$$

$$= \int \left(\frac{dy^3}{ds^2}\delta y + \frac{3ydy^2dx^2 + ydy^4}{ds^4} d\delta y - \frac{2ydy^3dx}{ds^4} d\delta x \right),$$

which, integrated by parts, becomes

$$\frac{3ydy^2dx^2 + ydy^4}{ds^4}\delta y - \frac{2ydy^3\delta x}{ds^4} + C +$$

$$\int \left(\frac{dy^3}{ds^2} - d\, \frac{3ydy^2dx^2 + ydy^4}{ds^4} \right) \delta y + \int d\, \frac{2ydy^3dx}{ds^4}\, \delta x = 0;$$

in which ds^2 is put for $dx^2 + dy^2$, and C is the arbitrary constant.

Because the equation must be satisfied so as to leave δy and δx under the sign $\int$ arbitrary, we must put their coefficients equal to naught, and shall thence get

$$\frac{dy^3}{ds^2} - d\, \frac{3ydy^2dx^2 + ydy^4}{ds^4} = 0 \quad \text{and} \quad d\, \frac{ydy^3dx}{ds^4} = 0;$$

consequently, the preceding variation reduces to

$$\frac{3ydy^2dx^2 + ydy^4}{ds^4}\delta y - \frac{2ydy^3dx}{ds^4}\delta x + C = 0.$$

If the extremities of the integral are given points, we have $\delta y = 0$, $\delta x = 0$, and thence $C = 0$; consequently, the conditions of the question are all satisfied.

To find the relation of x and y, it will clearly be sufficient to take the integral of

$$d\, \frac{ydy^3dx}{ds^4} = 0, \text{ which gives } \frac{ydy^3dx}{ds^4} = C' = \text{const.},$$

and to reject the other equation, or to put the δy under the sign $\int$, equal to naught.

The equation $\dfrac{y dy^3 dx}{ds^4} = C'$, gives

$$y = C' \times \frac{ds^4}{dy^3 dx} = C' \times \frac{(dx^2 + dy^2)^2}{dy^3 dx}$$

$$= C' \times \left(1 + \frac{dy}{dx}\right)^2 \times \frac{dx^4}{dx dy^3} = \frac{C'(1+p^2)^2}{p^3},$$

by putting $\dfrac{dy}{dx} = p$.

From $\quad y = \dfrac{C'(1+p^2)^2}{p^3} = C'(p^{-3} + 2p^{-1} + p),$

we get $\quad dy = C'(-3p^{-4} - 2p^{-2} + 1)\, dp,$

and thence $dx = \dfrac{dy}{p}$ becomes

$$dx = C'\left(-3p^{-5} dp - 2p^{-3} dp + \frac{dp}{p}\right),$$

whose integral gives

$$x = C'' + C'\left(\frac{3}{4p^4} + \frac{1}{p^2} + \text{h.l.}p\right),$$

in which C'' is the arbitrary constant, and h.l.p denotes the hyperbolic logarithm of p.

Supposing p to be eliminated from

$$y = \frac{C'(1+p^2)^2}{p^3}, \text{ and } x = C'' + C'\left(\frac{3}{4p^4} + \frac{1}{p^2} + \text{h.l.}p\right);$$

then, by putting in the resulting equation, the values of x and y at the given points at the extremities of the integral, we shall have two equations containing C' and C'' as unknowns, whose solutions will give the required values of the constants, as required; consequently, the required relation between x and y will be found.

Instead of supposing the extremities of the integral to be given, it will clearly be sufficient to use other conditions; such as will enable us to find the constants C′ and C″, and thence to get the values of x and y that may correspond to any assumed value of p.

Thus, if the limits of the variation of the integral are not given points; then, if the variation

$$\frac{3ydy^2dx^2 + ydy^4}{ds^4}\,\delta y - \frac{2ydy^3dx}{ds^4}\,\delta x + \mathrm{C} = 0,$$

is taken from the values of x and y represented by x' and y' to the values represented by x'' and y'', we shall have

$$\frac{3y''dy''^2dx''^2 + y''dy''^4}{ds''^4}\,\delta y'' - \frac{2y''dy''^3dx''}{ds''^4}\,\delta x''$$
$$-\left(\frac{3y'dy'^2dx'^2 + y'dy'^4}{ds'^4}\,\delta y' - \frac{2y'dy'^3dx'}{ds'^4}\,\delta x'\right) = 0.$$

If the co-ordinates at the extremities of the integral are independent of each other, it is manifest that this equation will be divided into the equations

$$\left(3 + \frac{dy''^2}{dx''^2}\right)\delta y'' - \frac{2dy''}{dx''}\,\delta x'' = 0,$$

and

$$\left(3 + \frac{dy'^2}{dx'^2}\right)\delta y' - \frac{2dy'}{dx'}\,\delta x' = 0,$$

which representing $\frac{dy}{dx}$ by p, are equivalent to

$$\frac{\delta y''}{\delta x''} = \frac{2p''}{3 + p''^2} \quad\text{and}\quad \frac{\delta y'}{dx'} = \frac{2p'}{3 + p'^2}.$$

Since

$$y = \frac{\mathrm{C}'(1 + p^2)^2}{p^3} \quad\text{and}\quad x = \mathrm{C}'' + \mathrm{C}'\left(\frac{3}{4p^4} + \frac{1}{p^2} + \mathrm{h.l.}\,p\right),$$

we may to these join the equations

$$y'' = \frac{C'(1+p''^2)^2}{p''^3},\ y' = \frac{C'(1+p'^2)^2}{p'^3},$$

$$x'' = C'' + C'\left(\frac{3}{4p''^2} + \frac{1}{p''^2} + \text{h.l.}p''\right),$$

$$x' = C'' + C'\left(\frac{3}{4p'^4} + \frac{1}{p'^2} + \text{h.l.}p'\right);$$

and, representing the equations of the limiting curves by

$$y'' = \phi(x'') \quad \text{and} \quad y' = \psi(x'),$$

we shall have $\dfrac{\delta y''}{\delta x''} = \phi'(x'')$ and $\dfrac{\delta y'}{\delta x'} = \psi(x')$,

which reduce the preceding equations to

$$\phi'(x'') = \frac{2p''}{3+p''^2} \quad \text{and} \quad \psi'(x') = \frac{2p'}{3+p'^2}.$$

Hence, we have eight equations, which will enable us to find the eight unknowns, x'', y'', p'', x', y', p', C', and C''; consequently, the points in which the curve represented by the equations

$$y = \frac{C'(1+p^2)^2}{p^3} \quad \text{and} \quad x = C'' + C'\left(\frac{3}{4p^4} + \frac{1}{p^2} + \text{h.l.}p\right)$$

intersects the limiting curves $y'' = \phi'(x'')$ and $y = \psi(x')$, may be supposed to have been found; and since the constants C' and C'' may be supposed to have been found, it clearly follows that the curve represented by

$$y = \frac{C'(1+p^2)^2}{p^3} \quad \text{and} \quad x = C'' + C'\left(\frac{3}{4p^4} + \frac{1}{p^2} + \text{h.l.}p\right),$$

may be supposed to be drawn, as required, between its limiting curves.

If for either limiting curve, as that whose co-ordinates are x' and y', we take the point whose co-ordinates are x' and y', then it is easy to perceive that our eight equations will be reduced to six, which will enable us to find the six unknowns, x'', y'', p'', p', C$'$, and C$''$, &c., as before.

REMARKS.—1. It is easy to perceive, from the solution of Ex. 19, at p. 113, that $\int \frac{ydy^3}{dx^2 + dy^2}$ represents the resistance of a solid of revolution around the axis of x, moving in a fluid of uniform density, in the direction of the axis of x with its smaller end foremost, whose nature we have determined, so as to make the resistance a minimum.

2. The example is substantially the same as that solved by Newton, at p. 120, vol. II., of his "Principia." If, in the preceding equations, we put $p = 1$, and y' for the corresponding value of y and $x = 0$, then

$$y' = 4C', \quad \text{or} \quad C' = \frac{y'}{4}, \quad 0 = C'' + \frac{7}{4}C', \quad \text{or} \quad C'' = -\frac{7C'}{4}.$$

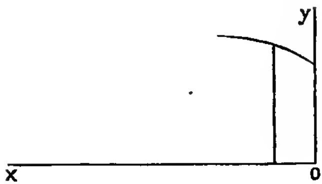
From the substitution of the values of the constants, the equations become

$$y = \frac{y'}{4}\frac{(1+p^2)^2}{p^3}, \quad \text{and} \quad x = \frac{y'}{4}\left(\frac{3}{4p^4} + \frac{1}{p^2} + \text{h. l.}\, p - \frac{7}{4}\right);$$

which clearly reduce to y' and 0 at the origin of the co-ordinates, since h. l. $1 = 0$. If we put $p = 0.9$, we readily get $y = 1.123\,y'$ and $x = 0.130\,y'$ very nearly, and $p = 0.8$ gives $y = 1.313\,y'$ and $x = 0.354\,y'$ nearly, and so on.

Hence, when the extremities of the integral are fixed points, as at p. 337, we easily perceive how the equation which connects y and x may be represented by linear description.

Thus, by putting $y' = 1$, and assuming ox and oy for the positive directions of the rectangular co-ordinates, having o for their origin, we set 1 from o on the axis of y for a point



in the curve, and then set 0.130 from o on the axis of x, through which (point) we erect the perpendicular 1.123 to the axis of x for the corresponding value of y, and then having set 0.354 for x' from o, as before, we draw the perpendicular through the point to the axis of x equal to 1.313 for the corresponding value of y, and so on to any required extent; then, a curve drawn with a steady hand through the points thus found will be such, that by revolving around the axis of x it will generate a solid, which, moving in a fluid from x toward o, it will meet with less resistance than any other solid, whose end diameters and height are the same. It is manifest, that the preceding construction is substantially the same as that of Newton.

4. To find the curve surface, whose area between given limits is a minimum.

Agreeably to what is shown at p. 302, the double integral

$$s = \iint dxdy \sqrt{\left\{1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right\}},$$

when taken between the proposed limits, may be taken to represent the required surface.

By taking the integral of the variation of the surface, we have

$$\delta s = \delta \iint dxdy \sqrt{\left\{1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right\}}$$

$$= \iint \delta dxdy \sqrt{\left\{1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right\}} = \iint \frac{p\delta p + q\delta q}{\sqrt{1 + p^2 + q^2}} dxdy,$$

by using p and q for $\frac{dz}{dx}$ and $\frac{dz}{dy}$, and because z is regarded as being a function of x and y, considered as being independent variables. Since

$$\delta p = \delta \frac{dz}{dx} = \frac{d\delta z}{dx} \quad \text{and} \quad \delta q = \frac{d\delta z}{dy}$$

on account of the constancy of dx and dy; then, if

$$P = \frac{p}{\sqrt{1+p^2+q^2}} \quad \text{and} \quad Q = \frac{q}{\sqrt{1+p^2+q^2}},$$

we shall have

$$\delta s = \iint P dy \frac{d\delta z}{dx} dx + \iint Q dx \frac{d\delta z}{dy} dy.$$

Hence, integrating by parts, we shall have

$$\delta s = \int P dy \delta z + \int Q dx \delta z - \iint \frac{dP}{dx} \delta z dx dy - \iint \frac{dQ}{dy} \delta z dx dy;$$

and it is clear that the part of this integral which is freed from one of the signs of integration, since it relates to the fixed limits, must be reduced to naught, since δz at the limits $= 0$. Hence, we shall have

$$\delta s = -\iint \frac{dP}{dx} \delta z dx dy - \iint \frac{dQ}{dy} \delta z dx dy,$$

which must equal naught, since s is to be a minimum; consequently, since δz, under the double sign of integration, is indeterminate or arbitrary, its factor $\frac{dP}{dx} + \frac{dQ}{dy}$, under the double sign of integration, must be reduced to naught, which gives $\frac{dP}{dx} + \frac{dQ}{dy} = 0$. By restoring the values of P and Q, and taking the indicated differential coefficients, the preceding equation will be reduced to its equivalent,

$$(1+q^2)\frac{dp}{dx} - pq\left(\frac{dQ}{dx} + \frac{dp}{dy}\right) + (1+p^2)\frac{dq}{dy} = 0;$$

consequently, if we put

$$\frac{dp}{dx} = r, \quad \frac{dq}{dx} = \frac{dp}{dy} = s, \quad \text{and} \quad \frac{dq}{dy} = t,$$

we shall have

$$(1 + q^2)\,r - 2pqs + (1 + p^2)\,t = 0,$$

for the equation of the partial differential coefficients of the sought surface, whose integral will, of course, be the surface.

REMARKS.—1. This example has been taken from p. 753, vol. II., of Lacroix's work, where it is remarked, in a foot note, that the equation

$$\frac{dP}{dx} + \frac{dQ}{dy} = 0 \quad \text{gives} \quad \frac{dP}{dx} = -\frac{dQ}{dy};$$

which is the condition of the immediate integrability of $Pdy - Qdx$; consequently, it is concluded that on the minimum surface, $\frac{pdy - qdx}{\sqrt{(1 + p^2 + q^2)}}$ is an exact differential, as well as $dz = pdx + qdy$. Thus, all plane surfaces will be found to satisfy these conditions; since

$$z = 0, \quad \frac{dz}{dx} = 0, \quad q = \frac{dz}{dy} = 0,$$

and of course the preceding conditions are reduced to naught; consequently, since the differentials of constants equal naught, it is manifest that the preceding differentials may be regarded as having constants for their exact integrals.

If these tests are applied to the surface whose equation is $az = xy$, they will be found to give

$$p = \frac{dz}{dx} = \frac{y}{a} \quad \text{and} \quad q = \frac{dz}{dy} = \frac{x}{a},$$

which reduce them to

$$\frac{ydy - xdx}{\sqrt{(a^2 + x^2 + y^2)}} \quad \text{and} \quad dz = \frac{ydx + xdy}{a} = \frac{d\,(xy)}{a};$$

consequently, since the first of these is not an exact differen tial, it follows that the proposed surface does not belong to the class of minimum surfaces. Nevertheless, if x and y are very small in comparison to a, it is clear that $\frac{ydy - xdx}{\sqrt{(a^2 + x^2 + y^2)}}$ does not sensibly differ from $\frac{ydy - xdx}{a}$, which is an exact differential; consequently, if a is very great, the surface $az = xy$ for finite values of x and y, will not greatly differ from a minimum surface.

2. Lacroix, at pages 806 and 875 of the volume cited, shows how to find the solid which, with a given capacity, contains the least surface.

Thus, since $\iint zdxdy$ and $\iint \sqrt{(1 + p^2 + q^2)}\, dxdy$

express the capacity and surface, and that the first is given, it is manifest if C stands for a constant, that when the surface is a minimum,

$$\iint Czdxdy + \iint \sqrt{(1 + p^2 + q^2)}\, dxdy =$$
$$\iint [Cz + \sqrt{(1 + p^2 + q^2)}]\, dxdy$$

will also be a minimum. Hence, using P and Q to stand for the same things as before, then taking x and y for the independent variables, we in like manner get the equation

$$C - \frac{dP}{dx} - \frac{dQ}{dy} = 0$$

or its equivalent,

$$C\,(1 + p^2 + q^2)^{\frac{3}{2}} - [(1 + q^2)\, r - 2pqs + (1 + p^2)\, t] = 0,$$

for the equation of partial differential coefficients of the required body, whose integral will, of course, represent the body.

Lacroix remarks, that the sphere and cylinder whose equations are represented by

$$z^2 + y^2 + x^2 = a^2, \quad \text{and} \quad z^2 + y^2 = a'^2,$$

will satisfy the preceding equations.

Thus, in the sphere $p = -\frac{x}{z}$, and $q = -\frac{y}{z}$,

which give $\sqrt{(1 + p^2 + q^2)} = \frac{a}{z}$,

and thence P and Q equal $-\frac{x}{a}$ and $-\frac{y}{a}$; which reduce the first of the preceding equations to $C + \frac{2}{a} = 0$, and in the cylinder the same equation is reduced to $C + \frac{1}{a'} = 0$.

5. To draw the shortest line possible from one point to another, on any proposed surface.

Let x, y, z, represent the rectangular co-ordinates of any point of the sought line; then, because the point is on a surface, z may be considered as being — a function of x and y regarded as being independent variables, and we shall have

$$dz = \frac{dz}{dx}dx + \frac{dz}{dy}dy = pdx + qdy.$$

From what is done at p. 240, we evidently have

$$ds = \sqrt{(dx^2 + dy^2 + dz^2)}$$

for the differential of the line, and

$$s = \int \sqrt{(dx^2 + dy^2 + dz^2)}$$

will represent the line, and its variation becomes

$$\delta s = \delta \int \sqrt{(dx^2 + dy^2 + dz^2)} = \int \delta \sqrt{(dx^2 + dy^2 + dz^2)}$$

$$= \int \left(\frac{dx}{ds}\, \delta dx + \frac{dy}{ds}\, \delta dy + \frac{dz}{ds}\, \delta dz \right)$$

$$= \int \left(\frac{dx}{ds}\, d\delta x + \frac{dy}{ds}\, d\delta y + \frac{dz}{ds}\, \delta dz \right)$$

$$= \frac{dx}{ds}\, \delta x + \frac{dy}{ds}\, \delta y + \frac{dz}{ds}\, \delta z -$$

$$\int \left(d\, \frac{dx}{ds}\, \delta x + d\, \frac{dy}{ds}\, \delta y + d\, \frac{dz}{ds}\, \delta z \right) + \mathrm{C};$$

C being the constant. Taking the integral from x', y', z', to x'', y'', z'', the constant will be removed, and we shall have

$$\delta s = \frac{dx''}{ds''}\, \delta x'' + \frac{dy''}{ds''}\, \delta y'' + \frac{dz''}{ds''}\, \delta z'' -$$

$$\left(\frac{dx'}{ds'}\, \delta x' + \frac{dy'}{ds'}\, \delta y' + \frac{dz'}{ds'}\, \delta z' \right) -$$

$$\int \left(d\, \frac{dx}{ds}\, \delta x + d\, \frac{dy}{ds}\, \delta y + d\, \frac{dz}{ds}\, \delta z \right).$$

Supposing the extremities of the integral to be fixed points, the part of the integral without the sign $\int$ will vanish; and since $\delta s = 0$, we must have

$$\int \left(d\, \frac{dx}{ds} + pd\, \frac{dz}{ds} \right) \delta x + \int \left(d\, \frac{dy}{ds} + qd\, \frac{dz}{ds} \right) \delta y = 0,$$

since $\delta z = p\delta x + q\delta y$.

Because δx and δy, under the sign $\int$, are arbitrary, their factors must be put equal to naught, which give

$$d\, \frac{dx}{ds} + pd\, \frac{dz}{ds} = 0 \quad \text{and} \quad d\, \frac{dy}{ds} + qd\, \frac{dz}{ds} = 0\,;$$

which are the equations of the minimum line, and are the same as those given by Lacroix, at p. 270 of the volume before cited.

Thus, to draw the shortest line possible from one point to another on the surface of a sphere whose equation is

$$r^2 = x^2 + y^2 + z^2.$$

Here $$dz = -\frac{x}{z}\,dx - \frac{y}{z}\,dy,$$

which gives $$p = -\frac{x}{z} \quad \text{and} \quad q = -\frac{y}{z},$$

which reduce the preceding equations of the minimum to

$$d\left(z\frac{dx}{ds} - x\frac{dz}{ds}\right) = 0 \quad \text{and} \quad d\left(z\frac{dy}{ds} - y\frac{dz}{ds}\right) = 0;$$

whose integrals may be expressed by

$$zdx + xdz = \text{A}ds, \quad \text{and} \quad zdy - ydz = \text{B}ds.$$

Multiplying the first of these by B and the second by A, we readily get $$\text{B}d\frac{x}{z} = \text{A}d\frac{y}{z};$$
whose integral gives

$$\frac{\text{A}y}{z} + \text{C} - \text{B}\frac{x}{z} = 0, \quad \text{or} \quad \text{A}y - \text{B}x + \text{C}z = 0;$$

which is the equation of a plane passing through the center of the sphere, and of course the shorter of the arcs of a great circle which passes from one of the given points to the other, is the required minimum distance.

REMARK.—Besides the minimum thus determined, which may be called the absolute minimum on the spheric surface, there is what may be called the relative maximum. For the lesser arc of the great circle, between the points being a minimum, the remaining arc of the same great circle will be the greatest distance on the surface between the points; supposing the distance to be measured in planes passing through the points.

(8.) We may now proceed to show how to distinguish between the maxima and minima in examples, but shall refer for this to Art. 876, p. 807, of the "Calcul Intégral" of Lacroix; noticing, that the maxima and minima can often be distinguished from each other by the nature of the case, as in the examples which have been given.

As we do not profess, in what has been done, to have given any thing more than the first principles of the Calculus of Variations, we must, for more ample details, refer to larger works: such as Woodhouse's "Treatise on the Calculus of Variations," and the "Calcul Intégral" of Lacroix, at p. 721. (See p. 614, Appendix.)

SECTION III.

INTEGRATION OF RATIONAL FUNCTIONS OF SINGLE VARIABLES, MULTIPLIED BY THE DIFFERENTIAL OF THE VARIABLE.

(1.) It is clear that such differentials must be of one of the two forms

$$(Ax^a + Bx^b + Cx^c + \&c.)\, dx,$$

and

$$\frac{Ax^a + Bx^b + Cx^c + \&c.}{A'x^{a'} + B'x^{b'} + C'x^{c'} + \&c.}\, dx;$$

in which the indices of x are supposed to be positive integers. Supposing the terms of these expressions to be arranged according to the descending or ascending powers of x, we may suppose the index of the highest power of x in the numerator of the fractional form to be less than the index of the highest power of x in the denominator; for if the index of x in the numerator is equal to or greater than in the denominator, it may be made less by arranging the terms of the numerator and denominator according to the descending powers of x, and then dividing the numerator by the denominator, when the fractional form will be reduced partly or wholly to the first of the preceding forms, accordingly as the numerator is not or is exactly divisible by the denominator.

(2.) By proceeding as in (9.) at p. 266, we may clearly suppose the integrals of all such differentials as the above to be found to any degree of exactness that may be required; which is clear from the circumstance mentioned by Newton, that they have the sums of the inscribed and circumscribed rectangles for their less and greater limits.

(3.) If we have differentials of the preceding forms, in which the indices of x are some of them positive fractions, by reducing the indices to their least common denominator, and representing unity divided by the least common denominator by $\frac{1}{p}$, and putting $y = x^{\frac{1}{p}}$, or $x = y^p$, we shall have $dx = py^{p-1}dy$; consequently, putting y for $x^{\frac{1}{p}}$ and $py^{p-1}dy$ for dx, the expressions will be changed into forms like to those at first supposed; and of course the integrals may be found to any degree of exactness, as before.

It is evident, if the first differential has any negative exponents, that their integrals may be found in algebraic forms, excepting when any of them happen to be -1, when the corresponding integral will be the hyperbolic logarithm of x multiplied by the corresponding coefficient of x^{-1}; and it is clear, that if in the fractional differential any of the indices of x are negative, they may be removed by multiplying the numerator and denominator by x with the same index taken with the positive sign, when it will follow, as before, that the integral can be found to the same degree of exactness, in the same manner as before.

(4.) It is manifest that the factor of dx, in the fractional differential, can be conceived to have been obtained from the addition of simpler fractions together, after having reduced them to a common denominator; consequently, the denominator will represent the common denominator of the fractions whose sum equals the proposed fraction. Hence, to find the component fractions, the first thing to be done is to resolve the denominator of the given fraction into factors, which can be done as follows.

Thus, by putting the proposed denominator equal to naught, we shall have an algebraic equation, whose roots, both real and imaginary (when the equal roots are included), will equal the number of units in the greatest exponent of x; noticing, that the imaginary roots always enter the equation in pairs of such forms, that the product of every two factors which give these roots will be real, or freed from their imaginary parts.

Hence, *we may suppose the denominator of the proposed fraction to consist of real, simple, and quadratic factors.* (See p. 440 of my Algebra, or most of the common works on that science.)

(5.) Having resolved the denominator into its factors, and taken any one of its unequal simple real factors for the denominator of any one of the component fractions, then we may assume a constant, to be found from the principles of identity of equations, for the numerator; since x must be of less dimensions in x in the numerator than in the denominator of the fraction.

To find the numerators of single quadratic factors taken for the denominators, they must generally consist of a constant term, and another constant for a factor of the simple power of x, observing that these constants are to be found, as before, on the principles of identity of equations.

When the proposed denominator contains a real, simple, or quadratic factor m times; then, if the numerators of the proposed fractions contain suitable dimensions in x, the fraction to be assumed must contain the mth power of the simple, real, or quadratic factor, and may contain all the lower powers of the same denominator for the denominators of other fractions, down to the simple or first power inclusive, provided x has suitable dimensions in the numerators of the proposed fractions; noticing, if x does not enter the numerator of the proposed fraction, that the fraction can not admit of any further reduction.

(6.) To illustrate what has been done, take the following simple

EXAMPLES.

1. To integrate the fraction $\dfrac{2x-5}{x^2-5x+6}\,dx.$

Putting the denominator equal to naught, we have the quadratic equation $x^2 - 5x + 6 = 0$; whose solution gives $x = 2$ or $x = 3$, and of course $x - 2$ and $x - 3$ are the factors of the proposed denominator.

Hence, agreeably to what is shown, we assume the proposed fraction equal to

$$\frac{A}{x-2}+\frac{B}{x-3},$$

and thence get the identical equation

$$\frac{2x-5}{x^2-5x+6}=\frac{A}{x-2}+\frac{B}{x-3}=\frac{(A+B)\,x-3A-2B}{x^2-5x+6},$$

which gives $\quad 2x - 5 = (A + B)\,x - 3A - 2B,$

which must be an identical equation; consequently, from equating the coefficients of like powers of x in the members of the equation, we get the equations $A + B = 2$ and $3A + 2B = 5$, which give $A = 1$ and $B = 1$, and thence the proposed differential is reduced to

$$\frac{2x-5}{x^2-5x+6}\,dx=\frac{dx}{x-3}+\frac{dx}{x-2}$$

whose integral is expressed by

$$\int \frac{2x - 5}{x^2 - 5x + 6}\, dx = \int \frac{dx}{x - 3} + \int \frac{dx}{x - 2} =$$

$\log (x - 3) + \log (x - 2) + \log C = \log C\,(x - 3)\,(x - 2)$, $\log C$ being the arbitrary constant (see p. 255); by putting $C = \frac{1}{6}$, we have $\int \frac{2x - 5}{x^2 - 5x + 6}\, dx = \log \frac{(x - 3).(x - 2)}{6}$, which commences with x.

2. To integrate $\frac{xdx}{(1 + x)^2}$ and $\frac{dx}{(1 + x)^3}$.

Because the denominator of the first of these differentials consists of two equal factors, $1 + x$ and $1 + x$, we assume

$$\frac{x}{(1 + x)^2} = \frac{A}{(1 + x)^2} + \frac{B}{1 + x} = \frac{A + B + Bx}{(1 + x)^2},$$

which gives $A = -B$ and $B = 1$; consequently, we shall have

$$\int \frac{xdx}{(1+x)^2} = -\int \frac{dx}{(1+x)^2} + \int \frac{dx}{1+x} = \frac{1}{1+x} + \log (1+x) + C.$$

Because x does not enter into the numerator of the differential $\frac{dx}{(1 + x)^3}$, we do not have any reduction like the preceding, and thence immediately get

$$\int \frac{dx}{(1 + x)^3} = -\frac{1}{2\,(1 + x)^2} + C$$

for the required integral.

Remark.—The reason for assuming

$$\frac{x}{(1 + x)^2} = \frac{A}{(1 + x)^2} + \frac{B}{1 + x},$$

becomes evident from dividing x by $x + 1$, which gives

$$\frac{x}{x + 1} = 1 - \frac{1}{1 + x},$$

and thence we shall have

$$\frac{x}{(1+x)^2} = -\frac{1}{(1+x)^2} + \frac{1}{1+x},$$

agreeably to the above assumption; and it is clear that a similar reasoning will be applicable in all analogous cases.

3. Integrate $2\int \frac{xdx}{1+x^2} = \log C\,(1+x^2)$, and reduce $\int \frac{a+bx+cx^2}{(x-e)^3}\,dx$ the expression to a proper form for integration, in order to get the true answer.

Thus, by putting $x-e=z$ or $x=e+z$, $dx=dz$, the form is

$$\int \frac{a+be+ce^2}{z^3}\,dz + \int \frac{2ce}{z^2}\,dz + \int \frac{c}{z}\,dz =$$

$$-\frac{a+be+ce^2}{2z^2} - \frac{2ce}{z} + \log Cz,$$

as required, C being the constant.

4. To integrate $\frac{7x-11}{x^3-2x^2-x+2}\,dx$.

Here, because the factors of the denominator are evidently $x-1$, $x+1$, and $x-2$, we assume

$$\frac{7x-11}{x^3-2x^2-x+2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x-2},$$

from which, as heretofore, by the method of undetermined coefficients, we may easily find the values of A, B, and C: we will here, however, use a modification of the method, which will often be preferable. Thus, to find A, we may suppose $x-1$ to differ insensibly from naught, which reduces the assumed equation to

$$\frac{7x-11}{x^3-2x^2-x+2} = \frac{A}{x-1},$$

on account of the comparative smallness of the other terms.

Dividing the denominators of this by $x - 1$, we have

$$\frac{7x - 11}{x^2 - x - 2} = A,$$

in which we must for x put 1, which gives $A = 2$. By putting $x + 1 =$ an infinitesimal, we, in like manner, have

$$\frac{7x - 11}{x^3 - 2x^2 - x + 2} = \frac{B}{x + 1},$$

whose denominators, divided by $x + 1$, give

$$\frac{7x - 11}{x^2 - 3x + 2} = B,$$

in which we must for x put -1, which gives $B = -3$ and in much the same way we get $C = 1$. Hence

$$\frac{7x - 11}{x^3 - 2x^2 - x + 2}\,dx = \frac{2dx}{x - 1} - \frac{3dx}{x + 1} + \frac{dx}{x - 2},$$

whose integral gives

$$\int \frac{7x - 11}{x^3 - 2x^2 - x + 2}\,dx$$

$$= 2\log(x - 1) - 3\log(x + 1) + \log(x - 2) + \log C$$

$$= \log C\,\frac{(x - 1)^2(x - 2)}{(x + 1)^3}.$$

REMARK.—The preceding method is clearly the same as to multiply by $x - 1$, divide the numerator and denominator in the first member by $x - 1$, and put $x - 1 = 0$ or $x = 1$ in the result, which will give the same value of A as before; and in like manner, by multiplying by $x + 1$ and $x - 2$ successively, we get B and C, the same as above.

5. To integrate $\dfrac{3 - x}{(x - 2)^3(x - 5)}\,dx$ and $\dfrac{(2 - x)^2}{(3 - x)^3}\,dx$.

Assuming

$$\frac{3 - x}{(x - 2)^3(x - 5)} = \frac{A}{(x - 2)^3} + \frac{B}{(x - 2)^2} + \frac{C}{x - 2} + \frac{D}{x - 5},$$

and supposing $x - 2$ to be an infinitesimal, we shall, on account of the comparative magnitudes of the terms, have

$$\frac{3-x}{(x-2)^3(x-5)}=\frac{A}{(x-2)^3};$$

or, dividing the denominators of these by $(x-2)^3$, we have $\frac{3-x}{x-5}=A$, which, since $x-2=0$, by putting 2 for x, gives $A=-\frac{1}{3}$. Hence, by subtracting

$$\frac{A}{(x-2)^3}=-\frac{1}{3}\,\frac{1}{(x-2)^3}$$

from the members of the assumed fractions, we have

$$\frac{3-x}{(x-2)^3(x-5)}+\frac{1}{3(x-2)^3}=-\frac{2}{3}\,\frac{1}{(x-2)^2(x-5)}$$

$$=\frac{B}{(x-2)^2}+\frac{C}{x-2}+\frac{D}{x-5};$$

consequently, proceeding with this in the same way as before, we shall have

$$B=-\frac{2}{3}\,\frac{1}{x-5}\quad\text{or}\quad(\text{since } x=2),\ B=\frac{2}{9}.$$

Subtracting $\frac{2}{9\,(x-2)^2}$ from the members of the preceding equation, we get

$$-\frac{2}{3}\,\frac{1}{(x-2)^2(x-5)}-\frac{2}{9}\,\frac{1}{(x-2)^2}=-\frac{2}{9}\,\frac{1}{(x-2)(x-5)}$$

$$=\frac{C}{x-2}+\frac{D}{x-5}.$$

Hence, as before, we have

$$C=-\frac{2}{9}\,\frac{1}{x-5}\quad\text{or}\quad(\text{since } x=2)\ C=\frac{2}{27};$$

consequently, subtracting $\frac{2}{27\,(x-2)}$ from the preceding equation, we, as before, get

$$-\frac{2}{9}\,\frac{1}{(x-2)(x-5)}-\frac{2}{27\,(x-2)}=-\frac{2}{27}\,\frac{1}{x-5},$$

which must equal the remaining fraction, and, of course,

$$D=-\frac{2}{27}.$$

Hence, from the substitution of the preceding values, the integral becomes

$$\int\frac{3-x}{(x-2)^3\,(x-5)}\,dx=\frac{1}{6\,(x-2)^2}-\frac{2}{9}\,\frac{1}{x-2}+\log C\left(\frac{x-2}{x-5}\right)^{\frac{2}{27}}.$$

In like manner, we have

$$\frac{(2-x)^2}{(3-x)^3}=\frac{1}{(3-x)^3}+\frac{B}{(3-x)^2}+\frac{C}{3-x},$$

which gives $B=-2$ and $C=1$; consequently, we shall have the integral

$$\int\frac{(2-x)^2}{(3-x)^3}\,dx=\frac{1}{2\,(3-x)^2}-\frac{2}{3-x}-\log C\,(3-x).$$

6. To find the integral of $\dfrac{x^3dx}{(x-a)^4}$.

Here we assume

$$\frac{x^3}{(x-a)^4}=\frac{A}{(x-a)^4}+\frac{B}{(x-a)^3}+\frac{C}{(x-a)^2}+\frac{D}{x-a},$$

or $$x^3=A+B\,(x-a)+C\,(x-a)^2+D\,(x-a)^3,$$

which must clearly be an ideutical equation; consequently, putting a for x, we get $A=a^3$, and, taking the differential of the members of the equation after dividing by dx, we have $$3x^2=B+2C\,(x-a)+3D\,(x-a)^2,$$

which, by putting a for x, reduces to $B=3a^2$. By taking the differentials of the members of the preceding equation, we have, after dividing by dx, $6x=2C+6D\,(x-a)$, which, by putting a for x, gives $C = 3a$; and taking the differentials again and proceeding as before, we have $D = 1$.

Hence, we shall have

$$\int \frac{x^3dx}{(x-a)^4} =$$

$$-\frac{a^2}{3(x-a)^3} - \frac{3a^2}{2(x-a)^2} - \frac{3a}{x-a} + \log(x-a) + C.$$

REMARK.—The method here used for finding the value of A, B, &c., appears to be of remarkable simplicity, and can clearly be applied in all analogous cases.

Otherwise, and more simply.—Put $x - a = z$ or $x = z + a$; then, since $x = z + a$, we have $dx = dz$, and thence $\int \frac{x^3dx}{(x-a^4)}$ is reduced to

$$\int \frac{(z+a)^3dz}{z^4} = \int \left(\frac{dz}{z} + \frac{3a}{z^2}dz + \frac{3a^2dz}{z^3} + \frac{a^3dz}{z^4}\right)$$

$$= \log z - \frac{3a}{z} - \frac{3a^2}{2z^2} - \frac{a^3}{3z^3} + C.$$

(7.) To complete the integration of rational fractional differentials, it clearly follows from what has been done, that it is necessary to reduce the integrals of differentials of the form $\frac{dz}{(z^2+b^2)^m}$ in which m is a positive integral greater than unity, to that of like form in which $m = 1$ Thus, from

$$\frac{dz}{(z^2+b^2)^{m-1}} = \frac{z^2dz}{(z^2+b^2)^m} + \frac{b^2dz}{(z^2+b^2)^m},$$

and $$d\frac{z}{(z^2+b^2)^{m-1}} = \frac{dz}{(z^2+b^2)^{m-1}} - \frac{2(m-1)z^2dz}{(z^2+b^2)^m},$$

by eliminating $\frac{z^2dz}{(z^2+b^2)^m}$ we get

$$\frac{(2m-3)\,dz}{(z^2+b^2)^{m-1}} + d\,\frac{z}{(z^2+b^2)^{m-1}} = \frac{2\,(m-1)\,b^2dz}{(z^2+b^2)^m};$$

or dividing by $2\,(m-1)\,b^2$, and taking the integrals of the quotients, we have

$$\int \frac{dz}{(z^2+b^2)^m} =$$

$$\frac{z}{2\,(m-1)\,b^2\,(z^2+b^2)^{m-1}} + \frac{2m-3}{2\,(m-1)\,b^2}\int \frac{dz}{(z^2+b^2)^{m-1}},$$

which reduces the proposed integral to that of

$$\int \frac{dz}{(z^2+b^2)^{m-1}};$$

and by changing m into $m-1$, we may in like manner reduce the integral

$$\int \frac{dz}{(z^2+b^2)^{m-1}} \quad \text{to that of} \quad \int \frac{dz}{(z^2+b^2)^{m-2}},$$

and so on to the integral of

$$\int \frac{dz}{z^2+b^2}, \quad \text{which equals} \quad \frac{1}{b}\tan^{-1}\frac{z}{b};$$

consequently, all the preceding integrals can be found, as required.

Thus, if $b=1$ and $m=2$, we shall have

$$\int \frac{dz}{(z^2+1)^2} = \frac{z}{2\,(z^2+1)} + \frac{1}{2}\tan^{-1} z + \mathrm{C},$$

C being the arbitrary constant. Also, if $b=1$ and $m=3$, we shall have

$$\int \frac{dz}{(z^2+1)^3} = \frac{z}{4\,(z^2+1)^2} + \frac{3}{4}\int \frac{dz}{(z^2+1)^2},$$

which, from

$$\int \frac{dz}{(z^2+1)^2} = \frac{z}{2(z^2+1)^2} + \frac{1}{2}\tan^{-1} z,$$

is reducible to

$$\int \frac{dz}{(z^2+1)^3} = \frac{z}{4(z^2+1)^2} + \frac{3z}{8(z^2+1)} + \frac{3}{8}\tan^{-1} z + C.$$

Otherwise.—Supposing the integral

$$\int \frac{dz}{z^2+b^2} = \int \frac{\frac{dz}{b}}{b\left(1+\frac{z^2}{b^2}\right)} = \frac{1}{b}\tan^{-1}\frac{z}{b}$$

to commence with z, then, by taking the differentials of the members of the equation, regarding b alone to be variable, we evidently get

$$-2bdb\int \frac{dz}{(z^2+b^2)^2} = -\frac{db}{b^2}\tan^{-1}\frac{z}{b} + \frac{1}{b}d\tan^{-1}\frac{z}{b};$$

or since $\qquad d\tan^{-1}\frac{z}{b} = -\frac{zdb}{b^2} \div \left(1+\frac{z^2}{b^2}\right),$

by substitution and dividing the members of the resulting equation by $-2bdb$, we shall get, after adding a constant, for correction,

$$\int \frac{dz}{(z^2+b^2)^2} = \frac{1}{2b^3}\tan^{-1}\frac{z}{b} + \frac{z}{2b^2(z^2+b^2)} + C.$$

It is evident that, by taking the differentials of the members of this equation, regarding b alone as variable, we may, in like manner, find the integral $\int \frac{dz}{(z^2+b^2)^3}$, and so on to any extent that may be required.

(8.) From what is said at p. 351, it is clear that if the differential of a variable contains terms which are affected with positive fractional exponents when the differential is of an integral form, or positive and negative exponents when the differential is of a fractional form, that the differentials may be changed into others in which the exponents shall be positive integers, or, as is usually said, the expressions may be rationalized.

Thus, the differential $(ax^{\frac{1}{3}} + bx^{\frac{1}{2}})\,dx$, which is of an integral form, by reducing the indices of x to a common denominator, is equivalent to $(ax^{\frac{2}{6}} + bx^{\frac{3}{6}})\,dx$; which, by putting $x = z^6$, and $dx = 6z^5dz$, is reduced to the integral form $6z^5\,(az^2 + bz^3)\,dz$, which is rationalized, or the exponents of z are integers. By taking the integral of the transformed differential, we shall have

$$\int (6az^7 + 6bz^8)\,dz = \frac{3}{4}\,az^8 + \frac{2}{3}\,bz^9 + C;$$

or, putting for z its value $x^{\frac{1}{6}}$, we have

$$\frac{3}{4}\,ax^{\frac{4}{3}} + \frac{2}{3}\,bx^{\frac{3}{2}} + C,$$

for the integral; C being the arbitrary constant.

Also, the integral $\int \frac{dx}{x^{\frac{1}{2}} - x^{\frac{1}{3}}}$ may clearly be rationalized by putting $x = z^6$ and $dx = 6z^5dz$, which will give

$$\int \frac{dx}{x^{\frac{1}{2}} - x^{\frac{1}{3}}} = 6\int \frac{z^3dz}{z-1} = 6\int\left(z^2dz + zdz + dz + \frac{dz}{z-1}\right)$$
$$= 2z^3 + 3z^2 + 6z + 6\log(z-1) + C;$$

which, since $z = x^{\frac{1}{6}}$, is easily reduced to

$$\int \frac{dx}{x^{\frac{1}{2}} - x^{\frac{1}{3}}} = 2\sqrt{x} + 3\sqrt[3]{x} + 6\sqrt[6]{x} + \log(x^{\frac{1}{6}} - 1) + C.$$

The integral $\int \frac{ax^{-\frac{1}{4}} + 1}{x^{\frac{1}{2}} + x^{\frac{7}{12}}}\,dx$, may be freed from the negative index of x in its numerator, by multiplying its numerator and denominator by $x^{\frac{1}{4}}$, which reduces it to

$$\int \frac{ax^{-\frac{1}{4}} + 1}{x^{\frac{1}{2}} + x^{\frac{7}{12}}}\, dx = \int \frac{a + x^{\frac{1}{4}}}{x^{\frac{3}{4}} + x^{\frac{5}{6}}}\, dx;$$

which, by putting $x = z^{12}$, becomes

$$12 \int \frac{a + z^3}{z^9 + z^{10}}\, z^{11} dz = 12 \int \frac{az^2 dz}{z+1} + 12 \int \frac{z^5 dz}{z+1}$$

$$= a\,[6z^2 - 12z + 12 \log (z+1)]$$

$$+ 12 \left(\frac{z^5}{5} - \frac{z^4}{4} + \frac{z^3}{3} - \frac{z^2}{2} + z - \log (z+1)\right) + C.$$

Also the integral $\int \frac{dx}{x^{\frac{1}{2}} + x^{\frac{1}{5}}}$, by putting $x = z^{10}$, becomes

$$10 \int \frac{z^9 dz}{z^5 + z^2} = 10 \int \frac{z^7 dz}{z^3 + 1} = 10 \int \left(z^4 dz - z dz + \frac{z dz}{z^3 + 1}\right)$$

$$= 2z^5 - 5z^2 + 10 \int \frac{z dz}{z^3 + 1};$$

noticing that this integral can be easily found by diverging series.

(9.) If the surds which enter into the differential coefficients of a given binomial form, contain the simple power of the variable, then it is clear that the differential may be rationalized in like manner as before.

Thus the differential $[3\,(a + bx)^{\frac{2}{3}} + 2\,(a + bx)^{\frac{1}{2}}]\, dx$ is rationalized by putting $a + bx = z^6$, which gives

$$dx = \frac{6z^5 dz}{b};$$

and thence the proposed differential becomes

$$(3z^4 + 2z^3) \times \frac{6z^5 dz}{b},$$

which is of a rational form, which reduces the integral of the proposed differential to

$$\frac{18z^{10}}{10b} + \frac{12z^{13}}{13b} + C = \frac{9z^{10}}{5b} + \frac{12z^{9}}{9} + \text{const.},$$

the same result that the immediate integration of the proposed differential will give.

Also the integral of $\frac{(a + bx)^{\frac{1}{2}} + (a + bx)^{\frac{1}{3}}}{(a + bx)^{\frac{1}{4}}} dx$, is easily rationalized by putting $a + bx = z^{12}$, which gives

$$dx = \frac{12z^{11}dz}{b};$$

and thence the proposed differential is reduced to the rational differential

$$\frac{12z^{17}dz}{bz^3} + \frac{12z^{15}dz}{bz^3},$$

whose integral is

$$\frac{4z^{15}}{5b} + \frac{12z^{13}}{13b} + C = \frac{4(a + bx)^{\frac{15}{12}}}{5b} + \frac{12(a + bx)^{\frac{13}{12}}}{13b} + C.$$

The integral of $\frac{adx}{x\sqrt{(a^2 - bx)}}$ is rationalized by putting $a^2 - bx = z^2$, which gives $dx = -\frac{2zdz}{b}$; and thence the proposed differential is reduced to the rational differential

$$\int \frac{adz}{z^2 - a^2} = \frac{1}{2}\int \frac{dz}{z - a} - \frac{1}{2}\int \frac{dz}{z + a},$$

whose integral is

$$\log \sqrt{\frac{z - a}{z + a}} + C, \quad \text{or} \quad \int \frac{2adz}{z^2 - a^2} = \log C\left(\frac{z - a}{z + a}\right)^{\frac{1}{2}},$$

as required.

The integral of $\frac{xdx}{\sqrt{(1+x)}}$ is rationalized by putting $1+x=z^2$, which gives $dx = 2zdz$ and $x = z^2 - 1$, which reduces the proposed differential to $2(z^2-1)\,dz$, whose integral is

$$2\left(\frac{z^3-3z}{3}\right)+\mathrm{C}.$$

The differential $\frac{xdx}{(1+x)^{\frac{1}{3}}}$, by putting $1+x=z^3$, is reduced to the rational differential $3z^4dz - 3zdz$, whose integral is $\frac{3z^5}{5}-\frac{3z^2}{2}+\mathrm{C}$, as required.

(10.) We now propose to show how to rationalize differentials whose coefficients involve the square root of an expression of the form $a+bx+cx^2$, or an expression that may be supposed to be comprehended by this form or come under it.

Thus, to rationalize the differential $\frac{dx}{\sqrt{a+bx+cx^2}}$, we assume

$$a+bx+cx^2=(x+z)^2c=x^2c+2xzc+z^2c,$$

which gives $a+bx=2xzc+z^2c$, and gives

$$z=\sqrt{\frac{a+bx+cx^2}{c}}-x=\frac{\sqrt{a+bx+cx^2}}{\sqrt{c}}-x,$$

and thence $x=\frac{a-cz^2}{2cz-b}$;

by adding z to x, we have

$$\sqrt{a+bx+cx^2}=\frac{(a-bz+cz^2)\,c}{(2cz-b)};$$

and by taking the differential of the value of x we also have

$$dx=\frac{-2c(a-bz+cz^2)\,dz}{(2cz-b)^2}.$$

Hence, from the substitution of these values in the given differential, it becomes

$$-\frac{2(a-bz+cz^2)\sqrt{c}dz}{(2cz-b)(a-bz+cz^2)};$$

or by reduction we have the differential $\frac{-2\sqrt{c}dz}{2cz-b}$, which is reduced to $-\frac{2cdz}{(2cz-b)} \div \sqrt{c}$, which, by integration, gives $-\frac{\log C(2cz-b)}{\sqrt{c}}$ for the integral $\int \frac{dx}{\sqrt{(a+bx+cx^2)}}$, as required.

If c is negative, or the proposed differential of the form $\frac{dx}{\sqrt{a+bx-cx^2}}$, we may find the factors of $a+bx-cx^2$ by solving the quadratic equation $a+bx-cx^2=0$, or its equivalent $x^2-\frac{b}{c}x=\frac{a}{c}$; whose roots will be found to be

$$x=\frac{b+\sqrt{(b^2+4ac)}}{2c} \quad \text{and} \quad x=\frac{b-\sqrt{(b^2+4ac)}}{2c},$$

the first being positive, and the second negative when a is positive.

Hence, if a' and b' stand for the first and second of these roots, we shall evidently, from well-known principles, have $(a'-x)(x-b')$ equal to $\frac{a}{c}+\frac{bx}{c}-x^2$; consequently, the proposed differential is reduced to $\frac{dx}{\sqrt{(a'-x)(x-b')}}$.

To rationalize this differential, we may assume

$$(a'-x)(x-b')=(x-b')^2z^2 \quad \text{or} \quad a'-x=(x-b')z^2,$$

which gives $$x=\frac{a'+b'z^2}{z^2+1};$$

whose differential gives

$$dx = \frac{2(b' - a')zdz}{(z^2 + 1)^2}.$$

Hence, since $\sqrt{(a' - x)(x - b')} = \left(\frac{a' - b'}{z^2 + 1}\right) z,$

we easily get $\dfrac{dx}{\sqrt{c}\sqrt{(a' - x)(x - b')}} = \dfrac{-2dz}{(z^2 + 1)}$; which is a rational differential, since z is not affected by the surd sign.

Remark.—If the proposed differential is of the form

$$\sqrt{(a + bx + cx^2)}\ dx,$$

by multiplying and dividing by $\sqrt{(a + bx + cx^2)}$ we have

$$\frac{(a + bx + cx^2)\ dx}{\sqrt{(a + bx + cx^2)}},$$

which is equivalent to

$$\frac{adx}{\sqrt{(a + bx + cx^2)}} + \frac{bxdx}{\sqrt{(a + bx + cx^2)}} + \frac{cx^2dx}{\sqrt{(a + bx + cx^2)}},$$

in which, as in the preceding examples, the irrationality is brought into the denominator of a fraction; which we may clearly always suppose to be done in practice.

To illustrate what has been done, take the following

EXAMPLES.

1. To find the integral of the differential $\dfrac{dx}{\sqrt{a + cx^2}}$.

Because the first of the preceding general forms, or the general form in (10), by putting $b = 0$, is reduced to the proposed example; it clearly follows, that by putting $b = 0$ in the results in (10), we shall get the corresponding results in the preceding example. Hence, we shall get $\frac{1}{\sqrt{c}}\log Cz$ for the integral $\int\frac{dx}{\sqrt{a + cx^2}}$, as required.

2. To integrate the differential $\frac{dx}{a + bx - x^2}$.

By putting 1 for c in the second of the preceding general forms (see p. 366), we shall have $x = \frac{a' + b'z^2}{z^2 + 1}$, a' and b' being the roots of the equation $x^2 - bx - a = 0$; and

$$\int\frac{dx}{\sqrt{(a + bx - x^2)}} = -2\tan^{-1}z + C,$$

or since (from p. 366), $z = \sqrt{\frac{a' - x}{x - b'}}$, we shall have

$$\int\frac{dx}{\sqrt{(a + bx - x^2)}} = C - 2\tan^{-1}\sqrt{\frac{a' - x}{x - b'}}.$$

3. To integrate the differential $\frac{dx}{\sqrt{(a^2 + c^2x^2)}}$.

It is manifest from the nature of a differential, that the integral of the differential $\frac{dx}{\sqrt{a^2 + c^2x^2}}$ in 3, must be expressed by a logarithm, and be of the form

$$\int\frac{dx}{\sqrt{(a^2 + c^2x^2)}} = \log\ (cx + \sqrt{a^2 + c^2x^2})$$

divided by c, or of the form $\log\ [cx + \sqrt{(a^2 + c^2x^2)}]$.

Since the differential of this is $cdx + \frac{c^2xdx}{\sqrt{(a^2 + c^2x^2)}}$, which divided by c, is easily reducible to

$$\frac{dx}{\sqrt{(a^2 + c^2x^2)}} \times (cx + \sqrt{a^2 + c^2x^2}),$$

which, divided by the quantity $cx + \sqrt{a^2 + c^2x^2}$, gives the proposed differential, as it ought to do. Consequently, after the addition of a constant to the preceding integral, it will represent the complete integral of the proposed differential, as required.

4. To find the integral of the differential $\dfrac{dx}{x\sqrt{(a^2 + c^2x^2)}}$.

Proceeding as in the last example, we have

$$\frac{dx}{x\sqrt{(a^2 + c^2x^2)}} = \frac{dx}{x^2} \div \sqrt{\left(\frac{a^2}{x^2} + c^2\right)},$$

or putting $\dfrac{1}{x} = y$, we get

$$\frac{dx}{x\sqrt{(a^2 + c^2x^2)}} = -\frac{dy}{\sqrt{(a^2y^2 + c^2)}},$$

or from 3, we have

$$\int \frac{dx}{x\sqrt{a^2 + c^2x^2}} = -\frac{1}{a} \log cx \frac{\sqrt{a^2 + c^2x^2} + a}{x},$$

as required. And thence

$$\int \frac{dx}{x\sqrt{(a^2 + c^2x^2)}} = -\log\frac{1}{a}\log c\sqrt{(a^2y^2 + c^2)} + ay.$$

5. To find the integral of $\dfrac{dx}{\sqrt{(1 - x^2)}}$ by rationalizing it.

It is manifest that, as in the Diophantine Analysis, we may assume

$$1 - x^2 = (1 - xz)^2 = 1 - 2xz + x^2z^2,$$

and thence get $x = \dfrac{2z}{1 + z^2}$, which gives

$$dx = \frac{2(1 - z^2)\,dz}{(1 + z^2)^2}, \quad \text{and} \quad \sqrt{(1 - x^2)} = 1 - xz = \frac{1 - z^2}{1 + z^2}.$$

Hence we shall get

$$\int \frac{dx}{\sqrt{(1-x^2)}} = 2\int \frac{dz}{1+z^2}$$

and $$\int \frac{dx}{\sqrt{(1-x^2)}} = 2\int \frac{dz}{1+z^2} = 2\tan^{-1} z + C.$$

Another form of the integral of the proposed differential is well known to be expressed by

$$\int \frac{dx}{\sqrt{(1-x^2)}} = \sin^{-1}x + C.$$

Since $z = \dfrac{1-\sqrt{1-x^2}}{x}$, it is clear from $\tan 2z = \dfrac{2z}{1-z^2}$, that we shall have $\tan 2z = \dfrac{x}{\sqrt{1-x^2}}$; consequently, we shall have $2\tan^{-1}z$ equal to $\sin^{-1}x$, which is as it ought to be.

The integral of $\dfrac{dx}{\sqrt{2bx-x^2}}$ is found by putting

$$\sqrt{2bx-x^2} = xz, \quad \text{or} \quad 2bx - x^2 = x^2z^2, \quad \text{or} \quad \frac{2b}{1+z^2} = x,$$

and thence $$dx = -\frac{4bzdz}{(1+z)^2}.$$

Hence, since $xz = \dfrac{2bz}{1+z^2}$, we get

$$\int \frac{dx}{\sqrt{2bx-x^2}} = -\int \frac{2dz}{1+z^2} = C - 2\tan^{-1}z,$$

as required.

Finally, the integrals of $\dfrac{dx}{x\sqrt{(1+x+x^2)}}$ and $\dfrac{dx}{\sqrt{(2ax+x^2)}}$ can be done in like manner to 4 and 3.

6. To find the integral of the differential $\dfrac{dx}{(1+x^2)^{\frac{3}{2}}}$.

This integral is $\displaystyle\int \frac{dx}{(1+x^2)^{\frac{3}{2}}} = \frac{x}{\sqrt{1+x^2}} + C$ (C = const);

since its differential is $\dfrac{dx}{(1+x^2)^{\frac{3}{2}}}$ the proposed differential.

Remarks.—It is easy to perceive that the differentials

$$\frac{dx}{\sqrt{(a+bx+cx^2)}} \quad \text{and} \quad \frac{dx}{\sqrt{(a+bx-cx^2)}}$$

(given at pp. 365 and 366) admit of the following useful transformations.

It is clear that we shall have

$$\frac{dx}{\sqrt{(a+bx+cx^2)}} = \frac{dy}{\sqrt{c}\sqrt{(A^2+y^2)}}$$

and
$$\frac{dx}{\sqrt{(a+bx-cx^2)}} = \frac{dz}{\sqrt{c}\sqrt{B^2-z^2}}.$$

It is clear that in this way the integrals

$$\int \frac{dx}{\sqrt{(1+x+x^2)}} \quad \text{and} \quad \int \frac{dx}{\sqrt{(1+x-x^2)}}$$

are reducible to the forms

$$\int \frac{dx}{\sqrt{\left\{\frac{3}{4}+\left(\frac{1}{2}+x\right)^2\right\}}} \quad \text{and} \quad \int \frac{dx}{\sqrt{\left\{\frac{5}{4}-\left(x-\frac{1}{2}\right)^2\right\}}};$$

which, by putting $x+\dfrac{1}{2}=z$, and $x-\dfrac{1}{2}=z'$, give

$$\int \frac{dz}{\sqrt{\left(\frac{3}{4}+z^2\right)}} = \log C\left\{z+\sqrt{\left(z^2+\frac{3}{4}\right)}\right\}$$

and $$\frac{2}{\sqrt{5}}\int\frac{\frac{\sqrt{5}}{2}\,dz'}{\sqrt{\left(\frac{5}{4}-z'^2\right)}}=\frac{2}{\sqrt{5}}\,\text{arc}\left(\text{rad}=\frac{\sqrt{5}}{2}\text{ and }\sin=z'\right).$$

Those correspond to the right members of the first and second equations reckoned downward; which are made integrable by putting

$$x+\frac{1}{2}=z \quad\text{and}\quad x-\frac{1}{2}=z',$$

as below.

(11.) Supposing the variable enters the coefficient of dx, in the form $x^m(a+bx^n)^p dx$, called a *binomial differential*, such that the exponents m, n, and p are integral or fractional, positive or negative; then we may clearly proceed to simplify its integral as follows:

1. It is manifest, that m and n may always be regarded as being integers, since they may always be reduced to integers by introducing a new variable.

2. n may be supposed to be always positive; for, by putting $\frac{1}{y}$ for x, x^n becomes $\frac{1}{y^n}=y^{-n}$, in which, when n is negative, $-n$ must, of course, be positive, and in y^{-n} the exponent is positive, as proposed.

3. Hence, the integral $\int x^m(a+bx^n)^p dx$ may always be supposed to be of the same general form, in which m and n are integers and n positive, while m and p may be negative, and at the same time p may be fractional.

4. Resuming the proposed differential, without regarding the preceding reductions, and putting $a+bx^n=z$, we shall

get $x = \left(\frac{z-a}{b}\right)^{\frac{1}{n}}$, which gives

$$dx = \frac{1}{nb^{\frac{1}{n}}}(z-a)^{\frac{1}{n}-1}dz;$$

consequently, the integral of the proposed differential is reduced to that of the integral

$$\frac{1}{nb^{\frac{m+1}{n}}}\int(z-a)^{\frac{m+1}{n}-1}z^p dz \dots\dots\dots (1);$$

which clearly shows that its integral *can be found when* $\frac{m+1}{n}$ *is an integer; this being called the condition of integrability of the expression.*

Because $\int x^m(a+bx^n)^p dx$ is equivalent to

$$\int x^{m+np}(ax^{-n}+b)^p dx,$$

if in this we put $ax^{-n}+b=z$ and proceed as before, the integral reduces to

$$-\frac{a^{\frac{m+1}{n}+p}}{n}\int(z-b)^{-\frac{m+1}{n}-p-1}z^p dz \dots\dots (2).$$

Hence, the integral can clearly be found, when *the condition of integrability* is represented by $\frac{m+1}{n}+p=$ an integer.

EXAMPLES.

1. To find the integral of $\int x(a+bx^2)^{\frac{1}{3}}dx$.

Here, since x^m is represented by x, we have $m=1$, and as bx^n is expressed by bx^2, we have $n=2$; consequently, the condition of integrability becomes $\frac{m+1}{n} = \frac{1+1}{2} = 1$ is satisfied, and the integral must be exact.

By substituting the values of m, n, and p, we shall have

$$\int x\,(a + bx^2)\,dx = \frac{1}{2b}\int (z - a)^{\circ}\, z^{\frac{1}{2}} dz$$

$$= [\text{since } (z-a)^{\circ} = 1]\ \frac{1}{2b} \times \frac{2}{3}\, z^{\frac{3}{2}} + \text{C}$$

$$= \frac{1}{3b}\, z^{\frac{3}{2}} + \text{C} = \frac{1}{3b}\,(a + bx^2)^{\frac{3}{2}} + \text{C}.$$

2. To find the integral $\int x^{-2}\,(a + bx^2)^{-\frac{1}{2}} dx$.

Since $m = -2$ and $n = 2$, we have

$$\frac{m+1}{n} + p = -\frac{1}{2} - \frac{1}{2} = -1 = \text{an integer},$$

which satisfies the second condition of integrability. Hence, we shall have

$$\int x^{-2}\,(a + bx^2)^{-\frac{1}{2}} dx = \int x^{-3}\,(ax^{-2} + b)^{-\frac{1}{2}}\, dx =$$

$$-\frac{a^{-1}}{2}\int (z - b)^{\circ}\, z^{-\frac{1}{2}} dz = -\frac{1}{a}\, z^{\frac{1}{2}} + \text{C} = -\frac{1}{ax}\sqrt{(a + bx^2)} + \text{C}.$$

3. To find

$$\int x^5\,(a + bx^3)^{\frac{2}{3}}\, dx = \frac{1}{3b^2}\int (z - a)\, z^{\frac{2}{3}}\, dz$$

$$= \frac{1}{3b^2}\left(\int z^3\, dz - a\int z^{\frac{2}{3}} dz\right) = \frac{1}{3b^2}\left(\frac{3}{8}\, z^{\frac{8}{3}} - \frac{3}{5}\, az^{\frac{5}{3}}\right) + \text{C}.$$

4. To find the integral $\int \frac{dx}{(1 + x^2)^{\frac{3}{2}}}$

Since the integral is equivalent to $\int (1 + x^2)^{-\frac{3}{2}} dx$, we have

$m = 0$, $\frac{m+1}{n} = \frac{1}{2}$, and as $p = -\frac{3}{2}$ we have $\frac{1}{2} - \frac{3}{2} = -1$ an integer, and we have

$$\int (1+x^2)^{-\frac{3}{2}} dx = -\int \frac{1}{2} (z-b)^{\circ} z^{-\frac{3}{2}} dz =$$

$$z^{-\frac{1}{2}} + C = \frac{x}{\sqrt{(1+x^2)}} + C.$$

5. To find the integral $\int \frac{x^5 dx}{x^2 + a^2}$.

Because the integral is equivalent to $\int x^5 (x^2 + a^2)^{-1} dx$,

we have $\quad m = 5, \quad n = 2, \quad$ and $\quad p = -1,$

and $\quad \frac{m+1}{n} = \frac{6}{2} = 3$ an integer,

and we have

$$\int x^5 (x^2 + a^2)^{-1} dx = \frac{1}{2} \int (z - a^2)^2 z^{-1} dz =$$

$$\frac{z^2}{4} - a^2 z + \frac{a^4}{2} \log z + C = \frac{x^4}{4} - \frac{a^2 x^2}{2} + \frac{a^4}{2} \log (a^2 + x^2) + C;$$

which may also be found by converting the fraction $\frac{x^5}{x^2 + a^2}$ into a series, arranged according to the descending powers of x, and then taking the integral of the quotient.

6. To find $\int \frac{x^5 dx}{\sqrt{(a + bx^2)}} = \int x^5 (a + bx^2)^{-\frac{1}{2}} dx.$

Since $m = 5$, $n = 2$, and $p = -\frac{1}{2}$, the equation

$$\int x^m (a + bx^n)^p dx = \frac{1}{nb^{\frac{m+1}{n}}} \int (z - a)^{\frac{m+1}{n} - 1} z^p dz$$

becomes

$$\int x^5 (a + bx^2)^{-\frac{1}{2}} dx = \frac{1}{2b^3} \int (z - a)^2 z^{-\frac{1}{2}} dz =$$

$$\frac{1}{2b^3} \left(\frac{2}{5} z^{\frac{5}{2}} - \frac{4}{3} a z^{\frac{3}{2}} + 2a^2 z^{\frac{1}{2}} \right) + C.$$

7. To find the integral $\int x^{-1}(a+bx^3)^{\frac{1}{3}}\,dx$.

This integral can clearly be easily found, since

$$\frac{m+1}{n}=\frac{-1+1}{3}=0,$$

and

$$\frac{1}{nb^{\frac{m+1}{n}}}\int(z-a)^{\frac{m+1}{n}-1}z^p dz$$

is reduced to

$$\frac{1}{3}\int(z-a)^{-1}z^p dz=\frac{1}{3}\int\frac{z^{\frac{1}{3}}dz}{z-a}.$$

By putting $z=y^3$ we have

$$dz=3y^2dy,\quad\text{and}\quad z^{\frac{1}{3}}dz=3y^3dy,$$

and thence we have

$$\frac{1}{3}\int\frac{z^{\frac{1}{3}}dz}{z-a}=\int\frac{y^3dy}{y^3-a}=\int\left(dy+\frac{ady}{y^3-a}\right)=y+\int\frac{ady}{y^3-a}.$$

By putting $y=va^{\frac{1}{3}}$, the integral $\int\frac{ady}{y^3-a}$ is reduced to $\int\frac{dv}{v^3-1}$; whose integral can be found from the principles at page 371, &c.

8. To find the integral $\int x^{-1}(a+bx^4)\,dx$.

Here $\int x^{-1}(a+bx^4)\,dx$ equals $a\log x+\frac{bx^4}{4}+C$, as required.

SECTION IV.

REDUCTIONS OF BINOMIAL DIFFERENTIALS TO OTHERS OF MORE SIMPLE FORMS.

(1.) THESE reductions generally result from the differential $dxy = ydx + xdy$, which gives

$$ydx = dxy - xdy \quad \text{and} \quad \int ydx = xy - \int xdy,$$

or

$$\int xdy = xy - \int ydx,$$

which is called *integration by parts;* and reduces the integral $\int ydx$ to the integral $\int xdy$, or the integral $\int xdy$ to $\int ydx$.

Thus, if we represent $(a + bx^n)^p$ by z^p, we shall have $(a + bx^n)^p = z^p$, and thence

$$(a + bx^n)^p x^m dx = z^p d\int x^m dx = z^p d\,\frac{x^{m+1}}{m+1},$$

which gives

$$\int x^m z^p dx = z^p \frac{x^{m+1}}{m+1} - \frac{pnb}{m+1}\int x^{m+n} z^{p-1} dx \ldots .(a);$$

which reduces the integral $\int x^m z^p dx$ to the integral $\int x^{m+n} z^{p-1} dx$, in which p is diminished by unity, while m is increased by n.

Also, from $x^m z^p dx = x^{m-n+1} d\int (a + bx^n)^p x^{n-1} dx$, we shall, as before, get

$$\int x^m z^p dx = z^{p+1}\frac{x^{m-n+1}}{(p+1)nb} - \frac{m-n+1}{(p+1)nb}\int x^{m-n}z^{p+1}dx \ldots (b);$$

which shows that the integral $\int x^m z^p dx$ is reduced to the integral $\int x^{m-n} z^{p+1} dx$.

Because the integrals in the right members of (a) and (b) admit of like changes, it clearly follows, if p is a positive integer greater than 1, while $m+1$, $m+n+1$, $m+2n+1$, &c., are finite, that the exponent p will finally, by successive applications of (a), be reduced to unity, and thence the integral $\int x^m (a+bx^n)^p dx$ will be determined; and in like manner, from (b), if p is a negative integer numerically greater than 1, while b, $p+1$, $p+2$, &c., are finite, it is manifest that the integral will be reduced, by successive applications of (b), to an integral in which $a+bx^n$ will enter in the form $(a+bx^n)^{-1} = \dfrac{1}{a+bx^n}$; consequently, agreeably to what has heretofore been shown, the integral will be reduced to the integral of a fractional differential, having a rational denominator, and is to be, according to what has been shown, regarded as known.

(2.) From $z = a + bx^n$, we get $a = z - bx^n$ and $b = zx^{-n} - ax^{-n}$, and thence

$$a\int x^m z^p dx = \int x^m z^{p+1} dx - b\int x^{m+n} z^p dx,$$

and $$b\int x^m z^p dx = \int x^{m-n} z^{p+1} dx - a\int x^{m-n} z^p dx.$$

Since, by putting $p+1$ for p in (a), it reduces

$$\int x^m z^{p+1} dx \text{ to } z^{p+1}\frac{x^{m+1}}{m+1} - \frac{(p+1)nb}{m+1}\int x^{m+n} z^p dx,$$

the first of the preceding equations gives

$$\int x^m z^p dx = z^{p+1}\frac{x^{m+1}}{a\,(m+1)} - \frac{(pn+n+m+1)b}{a\,(m+1)}\int x^{m+n}z^p dx \quad \ldots\ldots\ (c);$$

and, since by changing m into $m-n$, and p into $p+1$, in (a), it gives

$$\int x^{m-n}z^{p+1}dx = z^{p+1} \times \frac{x^{m-n+1}}{m-n+1} - \frac{(p+1)\,nb}{m-n+1}\int x^m z^p dx,$$

which being substituted for $\int x^{m-n}z^{p+1}dx$ in the second of the same equations, we readily get

$$\int x^m z^p\, dx =$$

$$z^{p+1} \times \frac{x^{m-n+1}}{(pn+m+1)\,b} - \frac{(m-n+1)\,a}{(pn+m+1)\,b}\int x^{m-n}z^p dx \ldots (d).$$

It will be perceived that in (c), the proposed integral is reduced to an integral in which m is changed into $m+n$, while in (d) it is changed into $m-n$. It is also clear that integrals which can not be reduced by (a) or (b), or with difficulty, can often be easily reduced by (c) or (d). Thus, the integral $\int x^{-2}(a^2+x^2)^{-1}dx$, in which $m=-2$, $n=2$, $p=-1$, $a=a^2$, and $b=1$, is by (c) immediately reduced to

$$\int x^{-2}(a^2+x^2)^{-1}dx =$$

$$z^{-1+1} \times \frac{x^{-2+1}}{a^2(-2+1)} - \frac{-2+2-2+1}{a^2(-2+1)} \times \int x^{-2+2}z^{-1}dx$$

$$= -\frac{1}{a^2x} - \frac{1}{a^2}\int\frac{dx}{a^2+x^2} = -\frac{1}{a^2x} - \frac{1}{a^4}\tan^{-1}\frac{x}{a} + \text{C}.$$

In like manner, by (d) the integral $\int x^3(a^2+x^2)^{-1}dx$ is easily reduced to

$$\int x^3(a^2+x^2)^{-1}dx = \frac{x^2}{2} - \frac{a^2}{2}\log(a^2+x^2) + \text{C}.$$

(3.) Multiplying the members of $z = a + bx^n$ by $x^m z^{p-1} dx$, and taking the integrals of the equal products, we have

$$\int x^m z^p dx = a \int x^m z^{p-1} dx + b \int x^{m+n} z^{p-1} dx.$$

To the products of the members of this by $\frac{pn}{m+1}$, adding the corresponding members of (a), we get

$$\left(1 + \frac{pn}{m+1}\right) \int x^m z^p dx = z^p - \frac{x^{m+1}}{m+1} + \frac{apn}{m+1} \int x^m z^{p-1} dx,$$

or its equivalent

$$\int x^m z^p dx = z^p \frac{x^{m+1}}{pn+m+1} + \frac{apn}{pn+m+1} \int x^m z^{p-1} dx \ldots (e);$$

which reduces the integral $\int x^m z^p dx$ to the integral $\int x^m z^{p-1} dx$, in which z^p is changed into z^{p-1} Thus, if $z = a^2 + x^2$ we have $p = 1$, $a = a^2$, and $n = 2$, and thence get

$$\int x^m z dx = z \frac{x^{m+1}}{m+3} + \frac{2a^2}{m+3} \int x^m dx$$

$$= (a^2 + x^2) \frac{x^{m+1}}{m+3} + \frac{2a^2}{m+3} \frac{x^{m+1}}{m+1} + C;$$

which is clearly the same result, that the immediate integral of the proposed differential will give. If p stands for a positive integer, it is clear that successive applications of the above formula will reduce $\int x^m z^p dx$ to

$$\int x^m z^{p-1} dx, \int x^m z^{p-2} dx \ldots \ldots \int x^m dx.$$

Changing p in (c) into $p + 1$, multiplying its members by $(p + 1) n + m + 1$, transposing, &c., we have

$$\int x^m z^p dx =$$

$$-z^{p+1}\frac{x^{m+1}}{a(p+1)n}+\frac{(p+1)n+m+1}{a(p+1)n}\int x^m z^{p+1}dx \ldots\ldots (f);$$

which reduces

$$\int x^m z^p dx \quad \text{to} \quad \int x^m z^{p+1}dx,$$

and
$$\int x^m z^{p+1}dx \quad \text{to} \quad \int x^m z^{p+2}dx,$$

and so on. Thus, if $p = -3$ we have

$$\int x^m z^{-3}dx = \frac{z^{-2}x^{m+1}}{2an} - \frac{-2n+m+1}{2an}\int x^m z^{-2}dx;$$

and then

$$\int x^m z^{-2}dx = \frac{z^{-1}x^{m+1}}{na} - \frac{-n+m+1}{na}\int x^m z^{-1}dx.$$

Hence, if $z = a^2 + x^2$ and m is a positive integer, the integral

$$\int x^m z^{-1}dx = \int \frac{x^m}{x^2+a^2};$$

which, by converting $\frac{x^m}{x^2+a^2}$ into a series arranged according to the descending powers of x, can now be easily found by the common methods of integration.

We will now, for convenience in what is to follow, collect the preceding formulas into a

(4.) *Table of Formulas for the Reduction of the Integral*

$$\int x^m (a+bx^n)^p dx = \int x^m z^p dx.$$

I.

$$\int x^m z^p dx = z^p \frac{x^{m+1}}{m+1} - \frac{pnb}{m+1}\int x^{m+n} z^{p-1}dx.$$

II.

$$\int x^m z^p dx = z^{p+1}\frac{x^{m-n+1}}{(p+1)nb} - \frac{m-n+1}{(p+1)nb}\int x^{m-n}z^{p+1}dx.$$

III.

$$\int x^m z^p dx = z^{p+1}\frac{x^{m+1}}{a(m+1)} - \frac{(pn+n+m+1)b}{a(m+1)}\int x^{m+n}z^p dx.$$

IV.

$$\int x^m z^p dx = z^{p+1}\frac{x^{m-n+1}}{(pn+m+1)b} - \frac{(m-n+1)a}{(pn+m+1)b}\int x^{m-n}z^p dx.$$

V.

$$\int x^m z^p dx = z^p\frac{x^{m+1}}{pn+m+1} + \frac{apn}{pn+m+1}\int x^m z^{p-1}dx.$$

VI.

$$\int x^m z^p dx = -z^{p+1}\frac{x^{m+1}}{a(p+1)n} + \frac{(p+1)n+m+1}{a(p+1)n}\int x^m z^{p+1}dx.$$

This table, under a different arrangement, is substantially the same as that of Mr. Young, at page 42 of his "Integral Calculus;" noticing, that our formula I. takes the place of his formula II., which is incorrect.

(5.) To perceive the use of the formulas, take the following

EXAMPLES.

1. To find the integral $\int x^{-4}(1-x^2)^{\frac{3}{2}}dx.$

Since $m=-4$, $n=2$, $p=\frac{3}{2}$, $a=1$, and $b=-1$, it is clear that formula I. reduces the integral to

$$\int x^{-4}z^{\frac{3}{2}}dx = z^{\frac{3}{2}}\frac{x^{-3}}{-3} - \int x^{-4+2}z^{\frac{3}{2}-1}dx$$

$$= -(1-x^2)^{\frac{3}{2}}\frac{1}{3x^3} - \int x^{-2}z^{\frac{1}{2}}dx;$$

and another application of I., reduces $\int x^{-2}(1-x^2)^{\frac{1}{2}}dx$ to

$$-(1-x^2)^{\frac{1}{2}}\frac{1}{x} - \int\frac{dx}{\sqrt{(1-x^2)}} = -\frac{\sqrt{(1-x^2)}}{x} - \sin^{-1}x + C;$$

hence, we have

$$\int x^{-4}(1-x^2)^{\frac{3}{2}}dx = -\frac{(1-x^2)^{\frac{3}{2}}}{3x^3} + \frac{(1-x^2)^{\frac{1}{2}}}{x} + \sin^{-1}x + C.$$

2. To find the integral $\int x^5(1-x^2)^{-3}dx = \int x^5z^{-3}dx$.

Since $m=5$, $n=2$, $p=-3$, $a=1$, and $b=-1$, from II., we have

$$\int x^5z^{-3}dx = \frac{z^{-2}x^4}{4} - \int x^3z^{-2}dx;$$

and another application of II. reduces

$$\int x^3z^{-2}dx \quad \text{to} \quad \int x^3z^{-2}dx = \frac{z^{-1}x^2}{2} - \int xz^{-1}dx.$$

Hence, $\displaystyle\int x^5z^{-3}dx = \frac{z^{-2}x^4}{4} - \frac{z^{-1}x^2}{2} + \int xz^{-1}dx\cdot$

since $\displaystyle\int xz^{-1}dx = \int\frac{xdx}{(1-x^2)} = -\frac{1}{2}\log(1-x^2) + C,$

the integral becomes

$$\int\frac{x^5dx}{(1-x^2)^3} = \frac{x^4}{4(1-x^2)^2} - \frac{x^2}{z(1-x^2)} - \frac{\log(1-x^2)}{2} + C.$$

3. To find the integral $\int x^{-4}(a^2+x^2)^{-1}dx$.

From $m=-4$, $n=2$, $p=-1$, $a=a^2$, and $b=1$, we get, from III.,

$$\int x^{-4}(a^2+x^2)^{-1}dx = \int x^{-4}z^{-1}dx = \frac{z^0x^{-3}}{-3a^2} - \frac{1}{a^2}\int x^{-2}z^{-1}dx$$

$$= -\frac{1}{3a^2x^3} - \frac{1}{a^2}\int x^{-2}z^{-1}dx;$$

and from another application of III., we have

$$\int x^{-2}z^{-1}dx = -\frac{x^{-1}}{a^2} - \int \frac{1}{a^2}z^{-1}dx = -\frac{1}{a^2x} - \int \frac{dx}{a^2(a^2+x^2)}.$$

Hence, we have

$$\int x^{-4}(a^2+x^2)^{-1}dx = -\frac{1}{3a^2x^3} + \frac{1}{a^4x} + \frac{1}{a^4}\int \frac{dx}{a^2+x^2},$$

and since

$$\frac{1}{a^4}\int \frac{dx}{a^2+x^2} = \frac{1}{a^5}, \quad \int \frac{\frac{dx}{a}}{1+\frac{x^2}{a^2}} = \frac{1}{a^5}\tan^{-1}\frac{x}{a};$$

consequently, we shall have

$$\int x^{-4}(a^2+x^2)^{-1}dx = -\frac{1}{3a^2x^3} + \frac{1}{a^4x} + \frac{1}{a^5}\tan^{-1}\frac{x}{a} + C.$$

4. To find the integral $\int x^5(a^2+x^2)^{-1}dx$.

From $m=5$, $n=2$, $p=-1$, $a=a^2$, and $b=1$, we shall, from IV., get

$$\int x^5(a^2+x^2)^{-1}dx = \int x^5z^{-1}dx$$

$$= \frac{x^4}{4} - \frac{a^2x^2}{2} + \frac{a^4}{2}\log(a^2+x^2) + C.$$

5. To find the integral $\int x^4(a^2+x^2)^{\frac{1}{2}}dx$.

From $m=4$, $n=2$, $p=\frac{1}{2}$, $a=a^2$, and $b=1$, V. gives

$$\int x^4 (a^2 + x^2)^{\frac{1}{2}} dx = \int x^4 z^{\frac{1}{2}} dx = z^p \frac{x^5}{5} + \frac{a^2}{6} \int x^4 z^{-\frac{1}{2}} dx.$$

From IV., we reduce

$$\int x^4 z^{-\frac{1}{2}} dx \quad \text{to} \quad x^4 z^{-\frac{1}{2}} dx = z^{\frac{1}{2}} \frac{x^3}{4} - \frac{3a^2}{4} \int x^2 z^{-\frac{1}{2}} dx,$$

and another application of the same formula reduces

$$\int x^2 z^{-\frac{1}{2}} dx \quad \text{to} \quad \int x^2 z^{-\frac{1}{2}} dx = z^{\frac{1}{2}} \frac{x}{2} - \frac{a^2}{2} \int z^{-\frac{1}{2}} dx;$$

consequently, since

$$\int z^{-\frac{1}{2}} dx = \int \frac{dx}{(a^2 + x^2)^{\frac{1}{2}}} = \log [x + \sqrt{(a^2 + x^2)}] + \log C$$

$$= \log C [x + \sqrt{(a^2 + x^2)}] = \log \left(\frac{x + \sqrt{(a^2 + x^2)}}{a} \right),$$

by putting $C = \frac{1}{a}$. Hence, we get

$$\int x^4 (a^2 + x^2)^{\frac{1}{2}} dx = \frac{x^5 (a^2 + x^2)^{\frac{1}{2}}}{6} +$$

$$\frac{a^2 x^3 (a^2 + x^2)^{\frac{1}{2}}}{6 \times 4} - \frac{3a^4 x}{6 \times 4 \times 2} (a^2 + x^2)^{\frac{1}{2}} + \frac{3a^6}{6 \times 4 \times 2} \log \left(\frac{x + \sqrt{(a^2 + x^2)}}{a} \right)$$

for the sought integral, supposed to commence with x.

6. To find the integral $\int x (a^2 - x^2)^{-\frac{3}{2}} dx = \int x z^{-\frac{3}{2}} dx.$

Since $m = 1$, $n = 2$, $p = -\frac{3}{2}$, $a = a^2$, and $b = -1$, we get, from VI.,

$$\int x z^{-\frac{3}{2}} dx = \frac{z^{-\frac{1}{2}} x^2}{a^2} - \frac{1}{a^2} \int x z^{-\frac{1}{2}} dx;$$

consequently, since

$$\int x z^{-\frac{1}{2}} dz = \int \frac{x dx}{\sqrt{(a^2 - x^2)}} = -\sqrt{(a^2 - x^2)} + C,$$

we shall have

$$\int xz^{-\frac{3}{2}}dx = \frac{x^2}{a^2\sqrt{(a^2-x^2)}} + \frac{\sqrt{(a^2-x^2)}}{a^2} + C',$$

in which $C' =$ the constant $-\dfrac{1}{a}$.

7. To find the integral $\int x^m (1-x^2)^{-\frac{1}{2}}dx$.

From $m=n$, $n=2$, $p=-\frac{1}{2}$, $a=1$, and $b=-1$, we readily get, from IV.,

$$\int x^m (1-x^2)^{-\frac{1}{2}}dx =$$

$$-\frac{x^{m-1}\sqrt{(1-x^2)}}{m} + \frac{m-1}{m}\int x^{m-2}(1-x^2)^{-\frac{1}{2}}dx.$$

Substituting successively the odd integers, 1, 3, 5, &c., for m, in this, we have

$$\int \frac{xdx}{\sqrt{(1-x^2)}} = -\sqrt{(1-x^2)} + C,$$

$$\int \frac{x^3dx}{\sqrt{(1-x^2)}} = -\frac{x^2\sqrt{(1-x^2)}}{3} + \frac{2}{3}\int \frac{xdx}{\sqrt{(1-x^2)}}, \text{ \&c.};$$

from which we readily obtain Mr. Young's results at p. 44 of his "Integral Calculus." And putting the even integers, 0, 2, 4, 6, &c., successively for m in the preceding formula, we in like manner get

$$\int \frac{dx}{\sqrt{(1-x^2)}} = \sin^{-1}x + C,$$

$$\int \frac{x^2dx}{\sqrt{(1-x^2)}} = -\frac{x\sqrt{(1-x^2)}}{2} + \frac{1}{2}\int \frac{dx}{\sqrt{(1-x^2)}},$$

and so on; from which Mr. Young's results, at p. 45 of his work, can easily be found.

8. To find the integral $\int x^{-m} (1-x^2)^{-\frac{1}{2}}dx$.

Since $m = -m$, $n = 2$, $p = -\frac{1}{2}$, $a = 1$, and $b = -1$, we have, from III.,

$$\int x^{-m}(1-x^2)^{-\frac{1}{2}}dx$$

$$= -\frac{\sqrt{(1-x^2)}\,x^{-m+1}}{m-1} + \frac{m-2}{m-1}\int x^{-m+2}(1-x^2)^{-\frac{1}{2}}dx$$

$$= -\frac{\sqrt{(1-x^2)}}{(m-1)\,x^{m-1}} + \frac{m-2}{m-1}\int \frac{dx}{x^{m-2}\sqrt{(1-x^2)}}.$$

If we put 3, 5, 7, &c., for m in this formula, we have

$$\int \frac{dx}{x^3\sqrt{(1-x^2)}} = \frac{-1}{2x^3(1-x^2)^{-\frac{1}{2}}} + \frac{1}{2}\int \frac{dx}{x(1-x^2)^{\frac{1}{2}}},$$

and $$\int \frac{dx}{x^5\sqrt{(1-x^2)^{\frac{1}{2}}}} = -\frac{\sqrt{(1-x^2)}}{4x^4} + \frac{3}{4}\int \frac{dx}{x^3\sqrt{(1-x^2)}},$$

and so on; and putting 0, 2, 4, &c., for m, we have

$$\int \frac{dx}{\sqrt{(1-x^2)}} = \sin^{-1}x,$$

$$\int \frac{dx}{x^2\sqrt{(1-x^2)}} = -\frac{\sqrt{(1-x^2)}}{x} + C,$$

$$\int \frac{dx}{x^4(1-x^2)^{\frac{1}{2}}} = -\frac{\sqrt{(1-x^2)}}{3x^3} + \frac{2}{3}\int \frac{dx}{x^2\sqrt{(1-x^2)}},$$

and so on, to any extent. The preceding results agree with those of Mr. Young at pp. 46, 47, and 48 of his "Integral Calculus;" noticing, that the integral

$$\int \frac{dx}{x\sqrt{(1-x^2)}} = -\log\frac{1+\sqrt{(1-x^2)}}{x} + C$$

can not be found by the preceding process.

9. To find the integral

$$\int \frac{x^m dx}{\sqrt{(2ax - x^2)}} = \int x^{m-\frac{1}{2}} (2a - x)^{-\frac{1}{2}} dx.$$

Here $m = m - \frac{1}{2}$, $n = 1$, $a = 2a$, $b = -1$, and $p = -\frac{1}{2}$, and thence, from IV.,

$$\int x^{m-\frac{1}{2}} (2a - x)^{-\frac{1}{2}} dx =$$

$$-\frac{(2a - x)^{\frac{1}{2}} x^{m-\frac{1}{2}}}{m} + \frac{(2m - 1)a}{m} \int \frac{x^{m-1} dx}{\sqrt{2ax - x^2}},$$

or

$$\int \frac{x^m dx}{\sqrt{2ax - x^2}} =$$

$$-\frac{x^{m-1}\sqrt{2ax - x^2}}{m} + \frac{(2m - 1)a}{m} \int \frac{x^{m-1} dx}{\sqrt{2ax - x^2}};$$

which clearly shows that by a sufficient number of repetitions of the process, the integral will be reduced to

$$\int \frac{dx}{\sqrt{2ax - x^2}} = \text{versin}^{-1} \frac{x}{a} + C.$$

10. To find the integral $\int x^m (a^2 + x^2)^{-p} dx.$

Since $n = 2$, $a = a^2$, and $b = 1$, we get, from VI.,

$$\int x^m z^{-p} dx$$

$$= \frac{(a^2 + x^2)^{-p+1} x^{m+1}}{2a^2 (p - 1)} - \frac{2(1 - p) + m + 1}{2a^2 (p - 1)} \int x^m z^{-p+1} dx$$

$$= \frac{x^{m+1}}{2a^2 (p-1)(a^2 + x^2)^{p-1}} + \frac{2p - 3 - m}{-2a^2 (p-1)} \int x^m z^{-p+1} dx;$$

which, by putting $m = 0$, reduces to

$$\int \frac{dx}{(a^2+x^2)^p} = \frac{x}{2a^2(p-1)(a^2+x^2)^{p-1}} + \frac{2p-3}{2a^2(p-1)}\int \frac{dx}{(a^2+x^2)^{p-1}};$$

which agrees with an equation previously found.

(6.) It is easy to perceive that we may apply the method of integrating by parts, to integrals of transcendental forms.

Thus, to find the integral of $X \log^n x dx$, in which X is a function of x, and n denotes the nth power of $\log x$, we have

$$\int X dx \log^n x = \log^n x \int X dx - \int \left(\int X dx \times n \log^{n-1} x \frac{dx}{x} \right);$$

or, by putting $\int X dx = X$, we have

$$\int X dx \log^n x = \log^n x \int X dx - \int \left(n \frac{X_1}{x} dx \log^{n-1} x \right);$$

which, by putting $\frac{X_1}{x} dx = dX_2$, gives

$$\int \log^n x X dx = \log^n x \int X dx - n \int dX_2 \log^{n-1} x \dots\dots (1).$$

If n is a positive integer, and

$$X dx = dX_1, \quad \frac{X_1}{x} dx = dX_2, \quad \frac{X_2}{x} dx = dX_3, \quad \&c.,$$

are exact differentials, it is clear that $\log^{n-1} x$ will finally be reduced to $\log^0 x = 1$, and thence the integral finally be reduced to an algebraic form. Thus,

$$\int x^2 \log x dx = \log x \times \frac{x^3}{3} - \frac{x^3}{9} + C;$$

$$\int x^3 dx \log^2 x = \frac{x^4}{4} \log^2 x - \frac{2}{4} \int x^3 \log x dx,$$

and

$$\int x^3 \log x dx = \frac{x^4}{4} \log x - \frac{1}{4} \int x^3 dx = \frac{x^4}{4} \log x - \frac{1}{4 \times 4} x^4 + C,$$

and thence we shall have

$$\int x^3 dx \log^2 x = \frac{x^4}{4} \log^2 x - \frac{1}{8} x^4 \log x + \frac{1}{32} x^4 + C.$$

In a similar way, we have

$$\int x^m dx \log^n x = \frac{x^{m+1}}{m+1}\log^n x - \frac{n}{m+1}\int x^m \log^{n-1} x dx,$$

$$\int x^m dx \log^{n-1} x = \frac{x^{m+1}}{m+1}\log^{n-1} x - \frac{n-1}{m+1}\int x^m \log^{n-2} x dx,$$

and so on to any extent, when m is different from -1, or when $m+1$ is different from naught.

Hence, when $m+1$ is different from naught, we shall, from the requisite substitutions in the first of these equations, get

$$\int x^m dx \log^n x = \frac{x^{m+1}}{m+1}\left(\log^n x - \frac{n}{m+1}\log^{n-1} x + \frac{n(n-1)}{(m+1)^2}\log^{n-2} x - \frac{n(n-1)(n-2)}{(m+1)^3}\log^{n-3} x + \&c.\right) + C.$$

If $m+1 = 0$, or $m = -1$, we have

$$\int x^m dx \log^n x = \int \frac{dx}{x}\log^n x = \frac{\log^{n+1} x}{n+1} + C.$$

We may, in much the same way as before, find the integral

$$\int \frac{x^m dx}{\log^n x} = \int x^m dx \log^{-n} x = \int x^{m+1} \log^{-n} x \frac{dx}{x} =$$

$$-\frac{x^{m+1}\log^{-n+1} x}{n-1} + \frac{m+1}{n-1}\int x^{m+1}\frac{dx}{x}\log^{-n+1} x = -\frac{x^{m+1}}{n-1}$$

$$\left(\log^{-n+1} x + \frac{m+1}{n-2}\log^{-n+2} x + \frac{(m+1)^2}{(n-2)(n-3)}\log^{-n+3} x + \ldots\right)$$

$$+\frac{(m+1)^{n-1}}{1.2.3\ldots(n-1)}\int \frac{x^m dx}{\log x};$$

in which it is clear that the integral $\int \frac{x^m dx}{\log x}$ can not be found by this method. If we put $x^{m+1} = z$, we shall have

$$x^m dx = \frac{dz}{m+1} \quad \text{and} \quad \log x = \frac{\log z}{m+1};$$

and of course $$\frac{x^m dx}{\log x} = \frac{dz}{\log z},$$

whose integral can be found by series.

Thus, since $d(\log x) = \dfrac{dx}{x}$, we shall have

$$\frac{d(\log x)}{\log x} = \frac{dx}{x \log x},$$

and thence $$\int \frac{dx}{x \log x} = \log \log x + \text{const.}$$

If (as is sometimes done) we write $\log^2 x$ for $\log \log x$, $\log^3 x$ for $\log \log \log x$, and so on, then we ought evidently to express the second, third, &c., powers of $\log x$ by such forms as $(\log x)^2$, $(\log x)^3$, &c. Hence, adopting this notation,

we shall have $$\int \frac{dx}{x \log x} = \log^2 x + \text{const.},$$

$$\int \frac{dx}{x \log x \log^2 x} = \int \frac{\dfrac{dx}{x \log x}}{\log^2 x} = \int \frac{d(\log^2 x)}{\log^2 x} = \log^3 x + C,$$

$$\int \frac{dx}{x \log x \log^2 x \log^3 x} = \log^4 x + C',$$

and so on. If we integrate $\dfrac{dz}{\log z}$ by parts, we shall have

$$\int \frac{dz}{\log z} = \frac{z}{\log z} - \int z d\left(\frac{1}{\log z}\right) = \frac{z}{\log z} + \int \frac{dz}{(\log z)^2}$$

$$= \frac{z}{\log z} + \frac{z}{(\log z)^2} - \int z d \frac{1}{(\log z)^2}$$

$$= \frac{z}{\log z} + \frac{z}{(\log z)^2} + \frac{2z}{(\log z)^3} - 2\int z d \frac{1}{(\log z)^3}$$

$$= \frac{z}{\log z} + \frac{z}{(\log z)^2} + \frac{2z}{(\log z)^3} + \frac{2.3z}{(\log z)^4} + \frac{2.3.4z}{(\log z)^5} + \text{\&c.} + \text{const.}$$

$$= \frac{z}{\log z}\left(1 + \frac{1}{\log z} + \frac{1.2}{(\log z)^2} + \frac{1.2.3}{(\log z)^3} + \text{\&c.}\right) \text{const.};$$

which is clearly not adapted to finding the integral $\int \frac{dz}{\log z}$ in converging terms, on account of its evident divergency.

REMARKS.—1. Because

$$\int \frac{dz}{\log z} = \int \frac{zdz}{z \log z} = z \log^2 z - \int \log^2 z dz,$$

we may clearly take this expression for $\int \frac{dz}{\log z}$.

2. If $u = \log z$ we shall have $z = e^u$, e being the base of hyperbolic logarithms, which gives $\frac{dz}{\log z} = \frac{e^u du}{u}$; or, since

$$e^u = 1 + u + \frac{u^2}{2} + \frac{u^3}{2.3} + \frac{u^4}{2.3.4} +, \&c.,$$

we shall have

$$\int \frac{dz}{\log z} = \int \frac{du}{u} + \int du + \frac{1}{2}\int udu + \frac{1}{2.3}\int u^2 du + \&c.$$

$$= \log u + u + \frac{u^2}{2^2} + \frac{u^3}{2.3^2} + \frac{u^4}{2.3.4^2} + \&c. + \text{const.}$$

From $u = \log z$, we have $\log u = \log \log z = \log^2 z$, &c., and thence get

$$\int \frac{dz}{\log z} = \log^2 z + \log z + \frac{(\log z)^2}{2^2} + \frac{(\log z)^3}{2.3^2} + \&c. + \text{const.}$$

(See pp. 55 and 56 of Young's "Integral Calculus.")

(7.) It is easy to perceive that we may readily find the integral of the differential, $x^m a^x dx$, which involves the exponential a^x, in much the same way as before.

Thus, since $x^m a^x dx = x^m d \int a^x dx = x^m d \frac{a^x}{\log a}$,

we get, by integrating by parts,

$$\int x^m a^x dx = \frac{x^m a^x}{\log a} - \frac{m}{\log a}\int x^{m-1} a^x dx,$$

$$\int x^{m-1} a^x dx = \frac{x^{m-1} a^x}{\log a} - \frac{m-1}{\log a}\int x^{m-2} a^x dx,$$

and so on. Hence, we shall have

$$\int x^m a^x dx = \frac{a^x}{\log a^x}\left(x^m - \frac{mx^{m-1}}{\log a} + \right.$$

$$\left.\frac{m(m-1)x^{m-2}}{(\log a)^2} - \frac{m(m-1)(m-2)x^{m-3}}{(\log a)^3} + \text{\&c.}\right) + C;$$

noticing, that if m is a positive integer, the last term within the parentheses will be $\pm \frac{1.2.3\ldots.m}{(\log a)^m}$, accordingly as m is an even or odd number. It is easy to perceive that we shall, in the same way, have

$$\int a^{-x} x^m dx =$$

$$-\frac{a^{-x}}{\log a}\left(x^m + \frac{mx^{m-1}}{\log a} + \frac{m(m-1)x^{m-2}}{(\log a)^2} + \ldots + \frac{1.2.3\ldots m}{(\log a)^m}\right) + C.$$

The integral $\int \frac{a^x dx}{x^m} = \int a^x x^{-m} dx$ is also easily found to be expressed by the form

$$\int \frac{a^x dx}{x^n} = -\frac{a^x}{(m-1)x^{m-1}}$$

$$\left(1 + \frac{\log a}{m-2}x + \frac{(\log a)^2}{(m-2)(m-3)} + \ldots + \frac{(\log a)^{m-2}}{(m-2)(m-3)\ldots 1}x^{m-2}\right) +$$

$$\frac{(\log a)^{m-1}}{(m-1)(m-2)\ldots.1}\int \frac{a^x dx}{x}.$$

If we expand a^x according to the Exponential Theorem (b), given at page 51, we shall have

$$a^x = 1 + \log ax + \frac{(\log a)^2 x^2}{1.2} + \frac{(\log a)^3 x^3}{1.2.3} +, \text{\&c.};$$

consequently, we shall thence get

$$\int \frac{a^x dx}{x} =$$

$$\log x + \log ax + \frac{(\log a)^2}{1.2}\frac{x^2}{2} + \frac{(\log a)^3}{1.2.3}\frac{x^3}{3} + \frac{(\log a)^4}{1.2.3.4}\frac{x^4}{4} + \text{\&c.} + C.$$

It is hence easy to perceive how we can find the integrals

$$\int x^m a^x dx \quad \text{and} \quad \int \frac{a^x dx}{x^m},$$

by means of the Exponential Theorem; by converting a^x into a series arranged according to the ascending powers of x.

It is manifest that, in this way, we shall get the integral

$$\int \frac{a^x dx}{1-x} = \int \left\{1 + (1 + \log a)x + \left(1 + \log a + \frac{(\log a)^2}{1.2}\right) x^2 + \left(1 + \log a + \frac{(\log a)^2}{1.2} + \frac{(\log a)^3}{1.2.3}\right) x^3 + \&c.\right\} dx =$$

$$x + (1 + \log a)\frac{x^2}{2} + \left(1 \log a + \frac{(\log a)^2}{1.2}\right)\frac{x^3}{3} + \&c. + \text{const.},$$

noticing, if $a = e$ the base of hyperbolic logarithms, since $\log e = 1$, we shall have

$$\int \frac{e^x dx}{1-x} = x + \left(1 + \frac{1}{1}\right)\frac{x^2}{2} + \left(1 + 1 + \frac{1}{1.2}\right)\frac{x^3}{3} + \&c. + \text{const.}$$

Because $\dfrac{dx}{1-x} = -\dfrac{-dx}{1-x} = -d[\log(1-x)]$, we easily get

$$\int \frac{a^x dx}{1-x} = -\int a^x d\,[\log(1-x)];$$

which, integrated by parts, by the successive use of the sign $\int$, gives

$$\int \frac{a^x dx}{1-x} = -a^x \log(1-x) + \log a \int a^x \log(1-x)\,dx;$$

and integrating again by parts, and so on, we shall have

$$\int \frac{a^x dx}{1-x} = -a^x \log(1-x)$$

$$+ \log a\, a^x \int \log(1-x)\,dx - (\log a)^2 a^x \int dx \int \log(1-x)\,dx$$

$$+ (\log a)^3 a^x \int dx \int dx \int \log(1-x)dx - \&c. + \text{const.}$$

Because $\quad -\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} +$, &c.,

the indicated integrations in the equation can be performed as required; and thence the integral will be found. Hence we shall have

$$\int \frac{a^x dx}{1-x} =$$

$$a^x \left\{ \left(x + \frac{x^2}{2} + \frac{x^3}{3} + \text{\&c.}\right) - \log a \left(\frac{x^2}{1.2} + \frac{x^3}{2.3} + \frac{x^4}{3.4} + \text{\&c.}\right) + \right.$$

$$\left. (\log a)^2 \left(\frac{x^3}{1.2.3} + \frac{x^4}{2.3.4} + \text{\&c.}\right) - (\log a)^3 \left(\frac{x^4}{1.2.3.4} + \text{\&c.}\right) + \text{\&c.} \right\}.$$

If for a^x we put $1 + \log ax + (\log a)^2 \frac{x^2}{1.2} +$, &c., in this, and perform the indicated multiplication, we shall easily get the value of $\int \frac{a^x dx}{1-x}$ found above.

Because $\int \frac{a^x dx}{1-x} = \int \frac{1}{1-x} d\int a^x dx = \int \frac{1}{(1-x)\log a} da^x$,

we shall, by integrating by parts, get

$$\int \frac{a^x dx}{1-x} = a^x \left(\frac{1}{(1-x)\log a} - \frac{1}{(1-x)^2(\log a)^2} + \right.$$

$$\left. \frac{1.2}{(1-x)^3(\log a)^3} - \frac{1.2.3}{(1-x)^4(\log a)^4} + \text{\&c.}\right) + \text{const.}$$

(See Lacroix "Calcul Intégral," p. 93.)

It is easy to perceive that the integral

$$\int \frac{e^x x dx}{(1+x)^2} = \frac{e^x}{1+x} + \text{C}.$$

For examples of integrals of the forms $\int \sin^m x \cos^n x dx$, &c., together with those of the form $\int \frac{\sin mx}{\cos nx} dx$, &c., we shall refer to La Croix "Calcul Intégral," pages 99, &c.; and to Young, on the same subject, pages 60, &c.; and for the exact Integrals of such expressions, see Young, pages 71, 75.

(8.) 1. To find the integrals

$$\int \frac{dx}{\sin^4 x \cos^3 x} \quad \text{and} \quad \int \frac{dx}{\sin^5 x \cos^2 x}.$$

By putting 4 and 3 for m and n, the first expression reduces to

$$\int \frac{dx}{\sin^4 x \cos^3 x} = -\frac{1}{3 \sin^3 x \cos^2 x} + \frac{5}{3}\int \frac{dx}{\sin^2 x \cos^3 x},$$

and putting 5 and 2 for m and n, the second reduces to

$$\int \frac{dx}{\sin^5 x \cos^2 x} = -\frac{1}{4 \sin^4 x \cos x} + \frac{5}{4}\int \frac{dx}{\sin^3 x \cos^2 x}.$$

In much the same way, we have

$$\int \frac{dx}{\sin^2 x \cos^3 x} = -\frac{1}{\sin x \cos^2 x} + 3\int \frac{dx}{\cos^3 x},$$

and $$\int \frac{dx}{\sin^3 x \cos^2 x} = \frac{1}{\sin^2 x \cos x} + 3\int \frac{dx}{\sin^3 x};$$

also, from (k) and (i) these results become

$$\int \frac{dx}{\cos^3 x} = \frac{\sin x}{2 \cos^2 x} + \frac{1}{2}\int \frac{dx}{\cos x},$$

and $$\int \frac{dx}{\sin^3 x} = -\frac{\cos x}{2 \sin^2 x} + \frac{1}{2}\int \frac{dx}{\sin x}.$$

Hence, from the requisite substitutions, we have

$$\int \frac{dx}{\sin^4 x \cos^3 x} =$$

$$-\frac{1}{3 \sin^3 x \cos^2 x} - \frac{5}{3 \sin x \cos^2 x} + \frac{5 \sin x}{2 \cos^2 x} + \frac{5}{2}\int \frac{dx}{\cos x},$$

and $$\int \frac{dx}{\sin^5 x \cos^2 x} =$$

$$-\frac{1}{4 \sin^4 x \cos x} + \frac{5}{4 \sin^2 x \cos x} - \frac{15 \cos x}{8 \sin^2 \cos x} + \frac{15}{8}\int \frac{dx}{\sin x};$$

consequently, the proposed integrals are reduced to the integrals $\int \frac{dx}{\cos x}$ and $\int \frac{dx}{\sin x}$.

2. To find the integrals

$$\int \frac{dx}{\sin x}, \quad \int \frac{dx}{\cos x}, \quad \text{and} \quad \int \frac{dx}{\sin x \cos x}.$$

Since $\cos x = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}$, we have

$$\int \frac{dx}{\cos x} = \int \frac{dx}{\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}} = 2\int \frac{\frac{dx}{2}}{\cos^2 \frac{x}{2}} \div \left(1 - \tan^2 \frac{x}{2}\right)$$

$$= \int \left\{ \frac{d \tan \frac{x}{2}}{1 + \tan \frac{x}{2}} - \frac{- d \tan \frac{x}{2}}{1 - \tan \frac{x}{2}} \right\}$$

$$= \log \frac{1 + \tan \frac{x}{2}}{1 - \tan \frac{x}{2}} + \text{const.}$$

$$= \log \cot \frac{1}{2}\left(\frac{\pi}{2} - x\right) + \text{const.}$$

Hence, since $\sin x = \cos\left(\frac{\pi}{2} - x\right)$, by changing x in the preceding results into $\frac{\pi}{2} - x$, then, since the differential of this arc is minus, we shall have

$$\int \frac{dx}{\sin x} = \log \tan \frac{x}{2} + \text{const.}$$

Because, $\sin^2 x + \cos^2 x = 1$, we shall have

$$\int \frac{dx}{\sin x \cos x} = \int \frac{\sin^2 x + \cos^2 x}{\sin x \cos x} dx = \int \frac{\sin x}{\cos x} dx + \int \frac{\cos x}{\sin x} dx$$

$$= \log \sin x - \log \cos x + \text{const.} = \log \tan x + \text{const.}$$

(9.) We now propose to show how to find the integrals of the differentials of a variable, into whose differential coefficients enter exponential and trigonometrical functions of the variable.

Thus, from

$$\int \mathrm{A}^{ax} \sin^m x dx = -\int \mathrm{A}^{ax} \sin^{m-1} x d \cos x,$$

by integrating by parts, we get

$$\int \mathrm{A}^{ax} \sin^m x dx = -\mathrm{A}^{ax} \sin^{m-1} x \cos x + \int \cos x d (\mathrm{A}^{ax} \sin^{m-1} x)$$

$$= -\mathrm{A}^{ax} \sin^{m-1} x \cos x + a \log \mathrm{A} \int \mathrm{A}^{ax} \sin^{m-1} x \cos x dx +$$

$$(m-1) \int \mathrm{A}^{ax} \sin^{m-2} x (1 - \sin^2 x)\, dx,$$

by putting $1 - \sin^2 x$ for $\cos^2 x$. From

$$\int \mathrm{A}^{ax} \sin^{m-1} x \cos x dx = \int \mathrm{A}^{ax} \frac{d (\sin^m x)}{m},$$

by integrating by parts, we have

$$\int \mathrm{A}^{ax} \sin^{m-1} x \cos x dx = \frac{\mathrm{A}^{ax} \sin^m x}{m} - \frac{a \log \mathrm{A}}{m} \int \mathrm{A}^{ax} \sin^m x dx.$$

Hence, from the substitution of this value and an obvious reduction, we get

$$\int \mathrm{A}^{ax} \sin^m x dx =$$

$$-\mathrm{A}^{ax} \sin^{m-1} x \cos x + \frac{a \log \mathrm{A}}{m} \mathrm{A}^{ax} \sin^m x -$$

$$\frac{(a \log \mathrm{A})^2}{m} \int \mathrm{A}^{ax} \sin^m x dx + (m-1) \int \mathrm{A}^{ax} \sin^{m-2} x dx -$$

$$(m-1) \int \mathrm{A}^{ax} \sin^m x dx;$$

or transposing so as to unite like integrals, we have

$$\int A^{ax}\sin^m x dx + (m-1)\int A^{ax}\sin^m x dx +$$

$$\frac{(a\log A)^2}{m}\int A^{ax}\sin^m x dx = \frac{(a\log A)^2+m^2}{m}\int A^{ax}\sin^m x dx =$$

$$\frac{A^{ax}\sin^{m-1}x}{m}(a\log A\sin x - m\cos x) +$$

$$(m-1)\int A^{ax}\sin^{m-2}x dx;$$

from which we get

$$\int A^{ax}\sin^m x dx = \frac{A^{ax}\sin^{m-1}x}{(a\log A)^2+m^2}(a\log A\sin x - m\cos x) +$$

$$\frac{m(m-1)}{(a\log A)^2+m^2}\int A^{ax}\sin^{m-2}x dx \quad \ldots\ldots \quad (a).$$

In like manner, we have

$$\int A^{ax}\cos^m x dx = \frac{A^{ax}\cos^{m-1}x}{(a\log A)^2+m^2}(a\log A\cos x + m\sin x) +$$

$$\frac{m(m-1)}{(a\log A)^2+m^2}\int A^{ax}\cos^{m-2}x dx \quad \ldots\ldots \quad (b).$$

If m is a positive integer greater than 2, it clearly follows that (a) and (b) will reduce the proposed integral to others in which the indices of $\sin x$ and $\cos x$ will be diminished by 2, or be less by 2 than in the proposed integrals.

EXAMPLES.

1. To find $\int e^x\sin^2 x dx$, and $\int e^x\cos^2 x dx$; $e =$ the hyperbolic base.

Since $A = e$, we have $\log A = \log e = 1$; and as $a = 1$ and $m = 2$, we have

$$\int e^x \sin x dx = \frac{e^x\sin x}{5}(\sin x - 2\cos x) + \frac{2}{5}e^x + \text{const.},$$

and $\int e^x \cos^2 x dx = \frac{e^x \cos x}{5} (\cos x + 2 \sin x) + \frac{2}{5} e^x + \text{const.}$

2. To find $\int 3^x \sin^2 x dx$, and $\int 3^x \cos^2 x dx$.

Here $A = 3$ and $\log A = \log 3 = 1.09861$, &c., $a = 1$ and $m = 2$; and thence, from (a) and (b),

$$\int 3^x \sin^2 x dx =$$

$$\frac{3^x \sin x}{(\log 3)^2 + 2^2} (\log 3 \sin x - 2 \cos x) + \frac{2}{(\log 3)^2 + 2^2} \times \frac{3^x}{\log 3},$$

and $$\int 3^x \cos^2 x dx =$$

$$\frac{3^x \cos x}{(\log 3)^2 + 2^2} (\log 3 \cos x + 2 \sin x) + \frac{2}{(\log 3)^2 + 2^2} \times \frac{3^x}{\log 3}.$$

3. To find $\int e^x \sin ax dx$, and $\int e^x \cos ax dx$.

From $$\int e^x \sin ax dx = \frac{1}{a} \int e^x \sin ax d(ax),$$

and $$\int e^x \cos ax dx = \frac{1}{a} \int e^x \cos ax d(ax);$$

by putting $ax = y$ or $x = \frac{y}{a}$, we have

$$\int e^x \sin ax dx = \frac{1}{a^2} \int e^{\frac{y}{a}} \sin y dy,$$

and $$\int e^x \cos ax dx = \frac{1}{a^2} \int e^{\frac{y}{a}} \cos y dy.$$

Hence, from (a) and (b), by putting $\frac{1}{a}$ for a, we have

$$\int e^{\frac{y}{a}} \sin y \frac{dy}{a^2} = \frac{e^{\frac{y}{a}}}{\left(\frac{1}{a}\right)^2 + 1^2} \left(\frac{\sin y}{a} - \cos y\right) \frac{1}{a} + \text{const.},$$

and $\int e^{\frac{y}{a}} \cos y \frac{dy}{a^2} = \frac{e^{\frac{y}{a}}}{\left(\frac{1}{a}\right)^2 + 1^2} \left(\frac{\cos y}{a} + \sin y\right) \frac{1}{a} + \text{const.}$

By re-substituting the value of y, we shall have

$$\int e^x \sin axdx = \frac{e^x}{1+a^2} (\sin ax - a \cos ax) + \text{const.},$$

and $\int e^x \cos axdx = \frac{e^x}{1+a^2} (\cos ax + a \sin ax) + \text{const.}$

4. To find $\int e^x \sin^3 xdx$, and $\int e^x \cos^3 xdx$.

From the tables given at pp. 77 and 78, we have

$$\sin^3 x = \frac{3 \sin x - \sin 3x}{4}, \quad \text{and} \quad \cos^3 x = \frac{3 \cos x + \cos 3x}{4},$$

which reduce the integrals to

$$\int e^x \sin^3 xdx = \frac{3}{4} \int e^x \sin xdx - \frac{1}{4} \int e^x \sin 3xdx,$$

and $\int e^x \cos^3 xdx = \frac{3}{4} \int e^x \cos xdx + \frac{1}{4} \int e^x \cos 3xdx.$

By taking the integrals by (a) and (b), agreeably to what has just been shown, we have

$$\int e^x \sin^3 xdx =$$

$$\frac{3e^x}{8} (\sin x - \cos x) - \frac{e^x}{40} (\sin 3x - 3 \cos 3x) + \text{const.},$$

and $$\int e^x \cos^3 xdx =$$

$$\frac{3e^x}{8} (\cos x + \sin x) + \frac{e^x}{40} (\cos 3x + 3 \sin 3x) + \text{const.}$$

(10.) We will now show how to find the integrals of differentials into whose differential coefficients enter arcs with algebraic functions of the arcs. Thus, to integrate

$$\int X (\sin^{-1} x)^n dx \quad \text{and} \quad \int X (\cos^{-1} x)^n dx,$$

X being an algebraic function of the arc, it is clear that we may put $\int X dx = X_1$, and thence, integrating by parts, get

$$\int X (\sin^{-1} x)^n dx = (\sin^{-1} x)^n X_1 - n \int (\sin^{-1} x)^{n-1} X_1 \frac{dx}{\sqrt{(1-x^2)}},$$

which, by putting $\int \frac{X_1 dx}{\sqrt{(1-x^2)}}$, gives

$$\int (\sin^{-1})^{n-1} X_1 \frac{dx}{\sqrt{(1-x^2)}} =$$

$$(\sin^{-1} x)^{n-1} X_2 - (n-1) \int (\sin^{-1} x)^{n-2} X_2 \frac{dx}{1-x^2};$$

and so on to any required extent in this, and such forms as

$$\int X (\cos^{-1} x)^n dx, \int X (\tan^{-1} x)^n dx, \int X (\cot^{-1} x)^n dx \ldots (A).$$

EXAMPLES.

1. To find $\int x \sin^{-1} x dx$ and $\int x \cos^{-1} x dx$.

Here $X = x$, and $X dx = \int x dx = \frac{x^2}{2}$, which gives $X_1 = \frac{x^2}{2}$; consequently,

$$\int X_1 \frac{dx}{\sqrt{(1-x^2)}} \quad \text{becomes} \quad \frac{1}{2} \int \frac{x^2 dx}{\sqrt{(1-x^2)}},$$

and thence

$$\int x \sin^{-1} X x = \frac{\sin^{-1} x X x^2}{2} - \frac{1}{2} \int x^2 (1-x^2)^{-\frac{1}{2}} dx =$$

$$\frac{x^2 \sin^{-1} x}{2} + \frac{1}{4} x \sqrt{(1-x^2)} - \frac{1}{4} \int \frac{dx}{\sqrt{(1-x^2)}} \cdot$$

$$= \frac{2x^2 - 1}{4} \sin^{-1} x + \frac{1}{4} x \sqrt{(1-x^2)} + \text{const.};$$

and in like manner,

$$\int x \cos^{-1} x dx = \frac{2x^2 + 1}{4} \cos^{-1} x - \frac{1}{4} x \sqrt{(1 - x^2)} + \text{const.}$$

2. To find $\int x^{m-1} \sin^{-1} x dx$ and $\int x^{m-1} \cos^{-1} x dx$.

Since $\int x^{m-1} dx = \frac{x^m}{m}$, we shall, by integrating by parts, from (A) get

$$\int x^{m-1} \sin^{-1} x dx = \frac{\sin^{-1} x_1 x^m}{m} - \frac{1}{m} \int x^m \frac{dx}{\sqrt{(-x^2)}},$$

and $\int x^{m-1} \cos^{-1} x dx = \frac{\cos^{-1} x_1 x^m}{m} + \frac{1}{m} \int x^m \frac{dx}{\sqrt{(1 - x^2}},$

we also have

$$\int x^m \frac{dx}{\sqrt{(1-x^2)}} = \frac{\sqrt{(1 - x^2)} x^{m-1}}{m} + \frac{m-1}{m} \int x^{m-2} \frac{dx}{\sqrt{(1-x^2)}},$$

and by the same process

$$\int x^{m-2} \frac{dx}{\sqrt{(1-x^2)}} = \frac{\sqrt{(1-x^2)} x^{m-3}}{m-2} - \frac{m-3}{m-2} \int x^{m-4} \frac{dx}{\sqrt{(1-x^2)}},$$

and so on to any required extent. It is hence clear, that if m is an odd positive integer, the complete integrals of the proposed integrals will be algebraic; while if m is an even positive integer, they will be reducible to circular arcs or be dependent on them.

REMARK.—It is easy to perceive that like conclusions are applicable to the integrals

$$\int x^{m-1} \tan^{-1} x dx \quad \text{and} \quad \int x^{n-1} \tan^{-1} x dx.$$

3. To find

$$\int\frac{\sin^{-1}x\,dx}{x^{m+1}}=\int\sin^{-1}x\,x^{-m-1}\,dx \quad\text{and}\quad \int\cos^{-1}x\,x^{-m-1}\,dx.$$

Because $\int x^{-m-1}\,dx = -\frac{x^{-m}}{m} = X_1$, these integrals become

$$\int\sin^{-1}x\,x^{-m-1}\,dx = -\frac{\sin^{-1}x_1 x^{-m}}{m}+\frac{1}{m}\int\frac{dx}{x^m\sqrt{(1-x^2)}},$$

and $$\int\frac{\cos^{-1}x\,dx}{x^{m+1}} = -\frac{\cos^{-1}x}{mx^m}-\frac{1}{m}\int\frac{dx}{x^m\sqrt{(1-x^2)}}.$$

If m is a positive integer not less than 2, we shall have

$$\int\frac{dx}{x^m\sqrt{(1-x^2)}}=\int x^{-m}(1-x^2)^{-\frac{1}{2}}\,dx=$$

$$-\frac{(1-x^2)^{\frac{1}{2}}x^{-m+1}}{m-1}+\frac{m-2}{m-1}\int x^{-m+2}(1-x^2)^{-\frac{1}{2}}\,dx,$$

by putting for n its value 2. By changing $-m$ into $-m+2$ we shall, in the same way, have

$$\int x^{-m+2}(1-x^2)^{-\frac{1}{2}}\,dx=$$

$$-\frac{(1-x^2)^{\frac{1}{2}}x^{-m+3}}{m-3}+\frac{m-4}{m-3}\int x^{-m+4}(1-x^2)^{-\frac{1}{2}}\,dx,$$

and so on. Hence, if m is an odd positive integer, we shall have $$\int\frac{dx}{x\sqrt{(1-x^2)}}=-\log\frac{1+\sqrt{(1-x^2)}}{x}+C,$$

which will enable us to find the integrals corresponding to any other odd positive integer, while it is manifest from what is done above, that when m is an even positive integer, the integral is algebraic, and can be exactly found by the preceding process. Thus

$$\int \sin^{-1} x_1 x^{-3} dx = -\frac{\sin^{-2} x_1 x^{-2}}{2} + \frac{1}{2}\int x^{-2}(1-x^2)^{-\frac{1}{2}} dx$$

$$= -\frac{\sin^{-1} x}{2x^2} - \frac{1}{2}\,\frac{(1-x^2)^{\frac{1}{2}}}{x} + \text{const};$$

and

$$\int \cos^{-1} x_1 x^{-3} dx = -\frac{\cos^{-1} x}{2x^2} + \frac{1}{2}\,\frac{(1-x^2)^{\frac{1}{2}}}{x} + \text{const.}$$

REMARK.—It is clear that we may, in much the same way,

find $\int \frac{\tan^{-1} x dx}{x^{m+1}}$ and $\int \frac{\cot^{-1} x dx}{x^{m+1}}$.

4. To find $\int (\sin^{-1} x)^2 dx$ and $\int x^2 (\cos^{-1} x)^2 dx$.

Since $X = 1$ we have $\int X dx = x$, and thence, from (A),

$$\int (\sin^{-1} x)^2 dx = (\sin^{-1} x)^2 x - 2 \int \sin^{-1} x, \quad \frac{x dx}{\sqrt{(1-x^2)}},$$

and in like manner from $X = x^2$, we have $\int x^2 dx = \frac{x^3}{3}$, and thence from (A),

$$\int x^2 (\cos^{-1} x)^2 dx = \frac{(\cos^{-1} x)^2 x^3}{3} + \frac{2}{3}\int \cos^{-1} x \frac{x^3 dx}{\sqrt{(1-x^2)}}.$$

We also have $\int \sin^{-1} x \frac{x dx}{\sqrt{(1-x^2)}}$

$$= -\sin^{-1} x \sqrt{(1-x^2)} + \int \sqrt{(1-x^2)} \frac{dx}{\sqrt{(1-x^2)}}$$

$$= -\sin^{-1} x \sqrt{(1-x^2)} + x;$$

consequently, we shall have

$$\int (\sin^{-1} x)^2 dx = (\sin^{-1} x)^2 x + 2 \sin^{-1} x \sqrt{(1-x^2)} - 2x + \text{const.}$$

From IV., at page 382, we have

$$\int x^3 \frac{dx}{\sqrt{(1-x^2)}} = \int x^3 (1-x^2)^{-\frac{1}{2}}\,dx$$

$$= -\frac{(1-x^2)^{\frac{1}{2}}\,x^2}{3} + \frac{2}{3}\int x\,(1-x^2)^{-\frac{1}{2}}\,dx$$

$$= -\frac{\sqrt{(1-x^2)}\,x^2}{3} - \frac{2\sqrt{(1-x^2)}}{3},$$

and thence $\displaystyle\int \cos^{-1} x x^3 \frac{dx}{\sqrt{(1-x^2)}}$

$$= -\cos^{-1} x\left(\frac{\sqrt{(1-x^2)}\,x^2}{3} + \frac{2}{3}\sqrt{(1-x^2)}\right) -$$

$$\int\left(\frac{\sqrt{(1-x^2)}\,x^2}{3} + \frac{2}{3}\sqrt{(1-x^2)}\right)\frac{dx}{\sqrt{(1-x^2)}}$$

$$= -\cos^{-1}x\left(\frac{\sqrt{(1-x^2)}\,x^2}{3} + \frac{2\sqrt{(1-x^2)}}{3}\right) - \frac{x^3}{9} - \frac{2x}{3} + \text{const.};$$

consequently,

$$\int x^2 (\cos^{-1} x)^2\,dx = \frac{(\cos^{-1} x)^2\,x^2}{3} -$$

$$\frac{2}{9}\cos^{-1} x\,[\sqrt{(1-x^2)}\,x^2 - 2\sqrt{(1-x^2)}\,] - \frac{x^4}{36} + \frac{2x}{36} + \text{const.}$$

5. To find $\int x \tan^{-1} x dx$ and $\int x \cot^{-1} x dx$.

Because $\int x dx = \frac{x^2}{2}$, we have

$$\int x\,(\tan^{-1}x)^2\,dx = \frac{(\tan^{-1}x)^2\,x^2}{2} - \int \tan^{-1}x\,\frac{x^2dx}{1+x^2}$$

and

$$\int x\,(\cot^{-1}x)^2\,dx = \frac{(\cot^{-1}x)^2\,x^2}{2} + \int \cot^{-1}x\,\frac{x^2dx}{1+x^2}.$$

If at IV., at p. 382, we put $a = b = 1$, $n = 2$, and $m = 2$, it will give

$$\int \frac{x^2 dx}{1 + x^2} = \int x^2 (1 + x^2)^{-1} dx = x - \int \frac{dx}{1 + x^2};$$

or taking the differentials,

$$\frac{x^2 dx}{1 + x^2} = dx - \frac{dx}{1 + x^2},$$

which can be found more simply by actual division.

Hence, by substitution, we have

$$\int x (\tan^{-1} x)^2 dx =$$

$$\frac{(\tan^{-1} x)^2 x^2}{2} - \int \tan^{-1} x dx + \int \tan^{-1} x \frac{dx}{1 + x^2},$$

and

$$\int x(\cot^{-1} x)^2 dx = \frac{(\cot^{-1} x)^2 x^2}{2} + \int \cot^{-1} x dx - \int \cot^{-1} x \frac{dx}{1 + x^2};$$

and since

$$\int \tan^{-1} x dx = \tan^{-1} x_1 x - \frac{1}{2} \log (1 + x^2),$$

and

$$\frac{dx}{1 + x^2} = d (\tan^{-1} x) = - d (\cot^{-1} x),$$

we shall have finally

$$\int x (\tan^{-1} x)^2 dx = \frac{(\tan^{-1} x)^2 x^2}{2} + \frac{(\tan^{-1} x)^2}{2} -$$

$$\tan^{-1} x_1 x + \frac{1}{2} \log (1 + x^2) + \tan^{-1} x \div 2 + \text{const.},$$

and

$$\int x (\cot^{-1} x)^2 dx = \frac{(\cot^{-1} x)^2 x^2}{2} + \frac{(\cot^{-1} x)^2}{2} +$$

$$\cot^{-1} x_1 x + \frac{1}{2} \log (1 + x^2) + \text{const.}$$

(See Lacroix "Calcul Intégral," pp. 95 and 96.)

REMARKS.—It is manifest, from what has been done, that to find an integral of the form

$$\int \frac{(a' + b' \cos z)\, dz}{(a + b \cos z)^n},$$

we ought to represent it by the form

$$\frac{A \sin z}{(a + b \cos z)^{n-1}} + \int \frac{(B + C \cos z)\, dz}{(a + b \cos z)^{n-1}}.$$

For by taking the differentials of these equals we have

$$\frac{(a' + b' \cos z)\, dz}{(a + b \cos z)^n} =$$

$$\frac{A \cos z dz}{(a + b \cos z)^{n-1}} + \frac{(n^{-1})\, Ab \sin^2 z dz}{(a + b \cos z)^n} + \frac{(B + C \cos z)\, dz}{(a + b \cos z)^{n-1}};$$

or, by omitting the common factor dz and a simple reduction, we have

$$a' + b' \cos z = A \cos z\, (a + b \cos z) + (n - 1)\, Ab\, (1 - \cos^2 z)$$
$$+ (B + C \cos z)\, (a + b \cos z),$$

or $\quad a' - (n - 1)\, Ab - Ba + (b' - Aa + Bb - Ca) \cos z$
$$- [Ab + (n - 1)\, Ab + Cb] \cos^2 z = 0,$$

which must clearly be an identical equation, and be satisfied so as to leave $\cos z$ and $\cos^2 z$ arbitrary; consequently, we must have

$$a' - (n - 1)\, Ab - Ba = 0, \quad b' - Aa - Bb - Ca = 0,$$
$$Ab - (n - 1)\, Ab + Cb = 0.$$

From the last of these equations we immediately get $C = (n - 2)\, A$, which reduces the second

$$b' - Aa - Bb - (n-2)\, Aa = 0, \quad \text{or} \quad b' - Bb - (n-1)\, A = 0,$$

which gives $\qquad B = \dfrac{b' - (n - 1)\, Aa}{b};$

and thence from the first equation we have

$$A = \frac{ab' - ba'}{(n-1)(a^2 - b^2)},$$

and of course

$$B = \frac{aa' - bb'}{a^2 - b^2} \quad \text{and} \quad C = \frac{(n-2)(ab' - ba')}{(n-1)(a^2 - b^2)}.$$

Hence, from the substitution of these values of A, B, C, in the assumed integral equation, we shall have

$$\int \frac{(a' + b' \cos z)\, dz}{(a + b \cos z)^n} = \frac{(ab' - ba') \sin z}{(n-1)(a^2 - b^2)(a + b \cos z)^{n-1}} + \frac{1}{(n-1)(a^2-b^2)} \int \frac{[(n-1)(aa'-bb') + (n-2)(ab'-ba') \cos z]}{(a + b \cos z)^{n-1}} dz$$

so that the complete integral is reduced to that of another in which n is represented by $n - 1$; consequently, if n is a positive integer greater than 1, we shall, by successive repetitions of the process, finally reduce the integral to that of an integral in which n is equal to unity, or to the form

$$\int \frac{(p + q \cos z)\, dz}{a + b \cos z}$$

(see Lacroix, p. 109); noticing that this integral is reducible by division to the more simple form

$$\frac{q^2}{b} + \frac{bp - aq}{b} \int \frac{dz}{a + b \cos z}.$$

If (with Lacroix, at p. 106) we put $\cos z = \dfrac{1 - x^2}{1 + x^2}$, we shall get

$$\int \frac{dz}{a + b \cos z} = 2 \int \frac{dx}{a + b + (a - b) x^2},$$

whose right member is clearly of an integrable form. Since

$$\cos z = \cos^2 \frac{z}{2} - \sin^2 \frac{z}{2},$$

we shall have

$$\int \frac{dz}{a + b\cos z} = \int \frac{dz}{a + b\left(\cos^2\frac{z}{2} - \sin^2\frac{z}{2}\right)}$$

$$= \int \frac{dz}{\cos^2\frac{z}{2}\left((a+b) + (a-b)\tan^2\frac{z}{2}\right)}$$

$$= \frac{2}{\sqrt{(a^2 - b^2)}}\int \frac{\sqrt{\frac{a-b}{a+b}}\,\frac{dz}{2} \div \cos^2\frac{z}{2}}{1 + \frac{a-b}{a+b}\tan^2\frac{z}{2}}$$

$$= \frac{2}{\sqrt{(a^2 - b^2)}}\tan^{-1}\sqrt{\frac{a-b}{a+b}}\tan\frac{z}{2} + \text{const},$$

in which a and b are supposed to be positive, a being greater than b. Since

$$2\tan^{-1}\sqrt{\frac{a-b}{a+b}}\tan\frac{z}{2} = \tan^{-1}\frac{\sin z\sqrt{(a^2 - b^2)}}{b + a\cos z},$$

we have

$$\int \frac{dz}{a + b\cos z} = \frac{1}{\sqrt{(a^2 - b^2)}}\tan^{-1}\frac{\sin z\sqrt{(a^2 - b^2)}}{b + a\cos z} + \text{const.};$$

noticing, that the same integral may also be expressed by either of the forms

$$\int \frac{dz}{a + b\cos z} = \frac{1}{\sqrt{(a^2 - b^2)}}\sin\ \frac{\sin z\sqrt{(a^2 - b^2)}}{a + b\cos z} + \text{const.},$$

$$\int \frac{dz}{a + b\cos z} = \frac{1}{\sqrt{a^2 - b^2}}\cos^{-1}\frac{b + a\cos z}{a + b\cos z} + \text{const.}$$

(11.) We have shown, at p. 262, that every differential expression containing a single variable, admits of an integral of either a diverging or converging form, by integrating by parts, as in *John Bernouilli's Theorem*. We have also applied, at the same page, the Theorem of Maclaurin to obtain series of more rapid convergency than can often be done by the aid of the Theorem of Bernouilli; and from the problem at p. 266 we have obtained formulas for the computation of such integrals by series of any degree of convergency that may be required.

Because, in what has been done, the series have been supposed to be arranged according to the ascending powers of the independent variable, we now propose to show how to apply series to find integrals when the series are arranged either according to the ascending or descending powers of the independent variable.

1st. To find the integral of a proposed differential by a series, it is manifestly necessary to convert the differential coefficient of the differential of the independent variable, according to the known methods, into a series arranged either according to the ascending or descending powers of the independent variable; then, to multiply the terms of the series by the differential of the (independent) variable, and to add an arbitrary constant to the sum of the integrals of the products, for the integral of the proposed differential.

It is manifest that the sum or generating function of the series thus found will be the finite integral of the proposed differential.

EXAMPLES.

1. To find the integral $\int \frac{dx}{1 + x^2}$ by a series, arranged either according to the ascending or descending powers of x.

By dividing dx by $1 + x^2$, when 1 is taken for the first term of the divisor, we have

$$\int \frac{dx}{1 + x^2} = \int (dx - x^2 dx + xdx^4 - xdx^6 + \&c.$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \&c. + \text{const.},$$

for the development when the series is arranged according to the ascending powers of the variable. And by taking x^2 for the first term of the divisor, we have

$$\int \frac{dx}{x^2+1} = \int \left(\frac{dx}{x^2} - \frac{dx}{x^4} + \frac{dx}{x^6} - \frac{dx}{x^8} + \text{\&c.} \right)$$

$$= -\frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \frac{1}{7x^7} - \text{\&c. const.,}$$

for the form of the integral, when it is arranged according to the descending powers of x. To find the constant, we remark, that x being the tangent of an arc whose radius $= 1$, it is clear that, supposing the arc and tangent to begin together, the constant in the first integral must equal naught, and the integral becomes

$$\int \frac{dx}{1+x^2} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} +, \text{\&c.};$$

while, by supposing x to be unlimitedly great in the second integral, it will clearly be reduced to the constant in its right member, since the terms which involve x must clearly be rejected on account of the infinite value of x, and at the same time $\int \frac{dx}{x^2+1}$ must equal $\frac{\pi}{2}$, the length of the arc of the quadrant of a circle whose radius $= 1$; consequently, the constant in the second integral equals $\frac{\pi}{2}$, and the integral becomes

$$\int \frac{dx}{x^2+1} = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \frac{1}{7x^7} -, \text{\&c.}$$

2. To find the integral $\int \frac{dx}{\sqrt{(1-x^2)}} = \sin^{-1} x$,

in a series arranged according to the ascending powers of x,

$$= x + \frac{1}{2.3} x^3 + \frac{1.3}{2.4.5} +, \text{\&c.};$$

which needs no correction, supposing the arc and sine to commence together.

Since the binomial theorem gives

$$\frac{1}{\sqrt{(1-x^2)}} = (1-x^2)^{-\frac{1}{2}} = 1 + \frac{x^2}{2} + \frac{1}{2}\cdot\frac{3}{4}x^4 + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}x^6 +, \&c.,$$

we shall have

$$\sin^{-1} x = x + \frac{1}{2.3}x^3 + \frac{1.3}{2.4.5}x^5 + \frac{1.3.5}{2.4.6.7}x^7 +, \&c.,$$

which needs no correction, supposing the arc and sine to begin together.

3. To express $\int \frac{dx}{\sqrt{(1+x^2)}} = \log\,[x + \sqrt{(1+x^2)}] + C,$ in a series.

Because

$$\frac{1}{\sqrt{(1+x^2)}} = 1 - \frac{1}{2}x^2 + \frac{1.3}{2.4}x^4 - \frac{1.3.5}{2.4.6}x^6 +, \&c.,$$

we shall have

$$\log\,[x + \sqrt{(1+x^2)}] = x - \frac{1}{2.3}x^3 + \frac{1.3}{2.4.5}x^5 - \frac{1.3.5}{2.4.6.7} +, \&c.,$$

which needs no correction, supposing the integral to commence with x; which we clearly may do, since $x = 0$ gives $\log 1 = 0$, as it ought to do.

4. To express $\int \frac{dx}{\sqrt{(x^2-1)}} = \log\,[x + \sqrt{(x^2-1)}] + C,$ in a series.

Since $\sqrt{(x^2-1)} = x\sqrt{\left(1-\frac{1}{x^2}\right)}$, we clearly have

$$\frac{1}{\sqrt{(x^2-1)}} = \frac{1}{x}\left(1-\frac{1}{x^2}\right)^{-\frac{1}{2}} =$$

$$\frac{1}{x} + \frac{1}{2x^3} + \frac{1.3}{2.4\,x^5} + \frac{1.3.5}{2.4.6\,x^7} +, \&c.;$$

consequently, we shall have

$$\log(x+\sqrt{x^2-1})+C=\log x-\frac{1}{2.2x^2}-\frac{1.3}{2.4.4x^4}-\frac{1.3.5}{2.4.6.6x^6}-, \&c.;$$

and by putting $x = 1$ in this, since $\log 1 = 0$, we have

$$C = -\frac{1}{2.2}-\frac{1.3}{2.4.4}-\frac{1.3.5}{2.4.6.6}-, \&c.,$$

and thence

$$\log(x+\sqrt{x^2-1})=\frac{1}{2.2}+\frac{1.3}{2.4.4}+\frac{1.3.5}{2.4.6.6}+\&c.$$

$$+\log x-\frac{1}{2.2x^2}-\frac{1.3}{2.4.4x^4}-\frac{1.3.5}{2.4.6.6x^6}+, \&c.$$

5. To convert $\int\frac{dx}{a+x}=\log(a+x)$ into a series.

Since $\frac{dx}{a+x}=\left(\frac{1}{a}-\frac{x}{a^2}+\frac{x^2}{a^3}-\frac{x^3}{a^4}+\&c.\right)dx$, we shall have $\log(a+x)=\frac{x}{a}-\frac{x^2}{2a^2}+\frac{x^3}{3a^3}-\frac{x^4}{4a^4}+\&c.+C$,

by putting $x = 0$ in this, we have $\log a = C$; consequently, we have

$$\log(a+x)=\log a+\frac{x}{a}-\frac{x^2}{2a^2}+\frac{x^3}{3a^3}-\frac{x^4}{4a^4}+, \&c.$$

REMARKS.—It is easy to perceive that this development can be immediately obtained from $\log a$, by changing a into $a + x$, and then developing $\log(a + x)$ according to the ascending powers of x, by Taylor's theorem.

6. To find the integral $\int\sqrt{\frac{1-e^2x^2}{1-x^2}}\,dx=\int\frac{\sqrt{1-e^2x^2}}{\sqrt{1-x^2}}\,dx$ in a series.

By converting $\sqrt{(1-e^2x^2)}$ into a series arranged according to the ascending powers of x, we have

$$\int\sqrt{\frac{1-e^2x^2}{1-x^2}}\,dx =$$

$$\int\frac{dx}{\sqrt{(1-x^2)}} - \frac{1}{2}e^2\int\frac{x^2dx}{\sqrt{1-x^2}} - \frac{1}{2.4}e^4\int\frac{x^4dx}{\sqrt{1-x^2}} -, \text{ \&c.}$$

Since $\int\frac{dx}{\sqrt{(1-x^2)}} = \sin^{-1}x$, and that formula IV., at p. 382, gives

$$\int\frac{x^2dx}{\sqrt{(1-x^2)}} = \int x^2(1-x^2)^{-\frac{1}{2}}dx = -\frac{x\sqrt{(1-x^2)}}{2} + \frac{1}{2}\sin^{-1}x,$$

and

$$\int\frac{x^4dx}{\sqrt{(1-x^2)}} = \int x^4(1-x^2)^{-\frac{1}{2}}dx$$

$$= -\left(\frac{x^3}{4} + \frac{1}{2}\cdot\frac{3}{4}x\right)\sqrt{(1-x^2)} + \frac{1}{2}\cdot\frac{3}{4}\sin^{-1}x,$$

and $\qquad \int x^6(1-x^2)^{-\frac{1}{2}}dx =$

$$-\left(\frac{x^5}{6} + \frac{1}{4}\cdot\frac{5}{6}x^3 + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}x\right)\sqrt{1-x^2} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\sin^{-1}x,$$

and so on; by collecting these results, we shall get

$$\int\sqrt{\frac{1-e^2x^2}{1-x^2}}\,dx = \sin^{-1}x + \frac{e^2}{2}\left(\frac{1}{2}x\sqrt{(1-x^2)} - \frac{1}{2}\sin^{-1}x\right) +$$

$$\frac{e^4}{2.4}\left\{\left(\frac{1}{4}x^3 + \frac{1}{2}\cdot\frac{3}{4}x\right)\sqrt{(1-x^2)} - \frac{1}{2}\cdot\frac{3}{4}\sin^{-1}x\right\} + \frac{1.3e^6}{2.4.6}$$

$$\left\{\left(\frac{x^5}{6} + \frac{1}{4}\cdot\frac{5}{6}x^3 + \frac{1.3.5}{2.4.6}x\right)\sqrt{(1-x^2)} - \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\sin^{-1}x\right\} +, \text{ \&c.},$$

for the integral; which needs no correction, supposing it to commence with x.

REMARKS.—It is easy to show that the preceding integral represents an arc of an ellipse, reckoned from the extremity of its minor axis.

For let $y = \frac{b}{a} \sqrt{(a^2 - x^2)}$ represent the common equation of an ellipse, then if e equals the ratio of the distance of the focus from the center to the semi-greater axis, we shall have $b = a \sqrt{(1 - e^2)}$ for the half minor axis, and the equation of the ellipse reduces to $y = \sqrt{(1 - e^2)} \sqrt{(a^2 - x^2)}$; whose differential gives $dy = -\frac{\sqrt{(1 - e^2)}\, x dx}{\sqrt{(a^2 - x^2)}}$. Hence

$$dy^2 + dx^2 = dz^2 = \frac{(x^2 - x^2 e^2)\, dx^2}{a^2 - x^2} + dx^2 \text{ or } dz = \sqrt{\frac{a^2 - e^2 x^2}{a^2 - x^2}}\, dx,$$

which agrees with the preceding differential equation when $a = 1$, and it is clearly the differential of an arc of the ellipse reckoned from the extremity of the minor axis.

If we put $x = a \sin \phi$ we shall have $dx = a \cos \phi d\phi$, which reduce the differential equation to

$$dz = a \sqrt{1 - e^2 \sin^2 \phi} \times d\phi;$$

if we put $a = 1$, the half major axis $= 1$, and we have

$$dz = \sqrt{1 - e^2 \sin^2 \phi} . d\phi;$$

or representing $\sqrt{1 - e^2 \sin^2 \phi}$ by Δ, we shall have

$$z = \int \Delta d\phi,$$

which is an elliptic function of the second kind, according to the notation of Legendre. (See p. 19 of his Exercises, "De Calcul Intégral.")

Since $\sin^2\phi = \dfrac{1-\cos 2\phi}{2}$, we have

$$1-e^2\sin^2\phi = 1-\frac{e^2}{2}+\frac{e^2}{2}\cos 2\phi = \left(1-\frac{e^2}{2}\right)\left(1+\frac{e^2}{2-e^2}\cos 2\phi\right),$$

or putting $2\phi=\theta$, we have $d\phi = \dfrac{d\theta}{2}$; consequently, we shall have

$$dz = \sqrt{(1-e^2\sin^2\phi)}\,d\phi = \sqrt{\frac{2-e^2}{8}} \times \sqrt{\left(1+\frac{e^2}{2-e^2}\cos\theta\right)}d\phi,$$

or putting $\dfrac{e^2}{2-e^2}=c$, we shall have

$$dz = \sqrt{\frac{2-e^2}{8}}\,\sqrt{1+c\cos\theta}\,d\theta.$$

By the binomial theorem,

$$\sqrt{(1+c\cos\theta)} = 1+\frac{1}{2}c\cos\theta - \frac{1}{8}c^2\cos^2\theta+\frac{1}{16}c^3\cos^3\theta - \frac{5}{128}c^4\cos^4\theta+\frac{7}{256}c^5\cos^5\theta -, \&c.;$$

consequently, multiplying the terms of this by $d\theta$, and taking the integrals from $\theta=0$ to $\theta=\pi$, we shall have

$$\int_0^\pi \sqrt{(1+c\cos\theta)}\,d\theta = \pi\left(1-\frac{1}{16}c^2-\frac{15}{1024}c^4-\frac{105}{16384}c^6-\&c.\right),$$

which gives the quadrantal arc of the ellipse, reckoned from the extremity of the minor axis,

$$= \sqrt{\frac{2-e^2}{2}} \times \frac{\pi}{2}$$

$$\left(1-\frac{1}{16}\frac{e^4}{(2-e^2)^2}-\frac{15}{1024}\frac{e^8}{(2-e^2)^4}-\frac{105}{16384}\frac{e^{12}}{(2-e^2)^6}-\&c.\right).$$

If we apply this formula to find the perimeter of the ellipse, whose major and minor axes are 12 and 9, we shall have

$$e^2 = 1 - \frac{3^2}{4^2} = \frac{7}{16} = 0.4375,$$

and thence

$$\frac{e^2}{2 - e^2} = 0.28;$$

hence, we easily find 0.99501, for the sum of the first three terms, within the parentheses, of the preceding series; and since

$$\sqrt{\frac{2 - e^2}{2}} = \sqrt{0.78125} = 0.88388,$$

we have

$$0.88388 \times 0.99501 = 0.87947.$$

This is the same result that Mr. Young has obtained at p. 116 of his "Integral Calculus," from the formula at p. 415, when taken between the same limits, or from $x = 0$ to $x = 1$, which gives

$$\frac{\pi}{2}\left(1 - \frac{1.1}{2.2}e^2 - \frac{1.1.1.3}{2.2.4.4}e^4 - \frac{1.1.1.3.3.5}{2.2.4.4.6.6}e^6 - \&c.\right)$$

for the length of the quadrantal arc of an ellipse when its half major axis is denoted by 1 or unity. Mr. Young obtained his result by calculating the first eight terms of this series, whereas the first three terms of our series have given the same result, which clearly shows that our formula is far more convergent than the preceding formula, which is the formula commonly used for the computation of the elliptic quadrant. It is easy to perceive that we shall have

$$\frac{355}{113} \times 24 \times 0.87947 = 66.31032$$

for the perimeter of the ellipse, which differs but little from Mr. Young's result.

(12.) We now propose to show how to apply series to the computation of integrals of the forms

$$\int X^m \, dx^m, \quad \int x^n \, dx^n, \quad \&c.,$$

which have been partially considered at pp. 312 to 315.

1. Thus, to find $\int \frac{dx^2}{1-x^2}$, we convert $\frac{1}{1-x^2}$ into a series,

$$\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 +, \&c.,$$

and thence get

$$\int^2 \frac{dx^2}{1-x^2} = \int dx \int (dx + x^2\,dx + x^4\,dx + x^6\,dx + \&c.)$$

$$= \int \left(x dx + \frac{x^3}{3}\,dx + \frac{x^5}{5}\,dx + \&c. + C dx \right)$$

$$= \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{30} + \&c. + Cx + C';$$

we also have

$$\int^3 \frac{dx^3}{\sqrt{1-x^2}} = \int dx \int dx \int dx \left(1 + \frac{1}{2}x^2 + \frac{1.3}{2.4}x^4 + \frac{1.3.5}{2.4.6}x^6 + \&c.\right)$$

$$= \int dx \int dx \left(x + \frac{1.1}{2.3}x^3 + \frac{1.3.1}{2.4.5}x^6 + \frac{1.3.5.1}{2.4.6.7}x^7 + \&c. + C\right)$$

$$= \int dx \left(\frac{x^2}{2} + \frac{1.1.1}{2.3.4}x^4 + \frac{1.3.1.1}{2.4.5.6}x^6 + \frac{1.3.5.1.1}{2.4.6.7.8}x^8 + \&c. + Cx + C'\right)$$

$$= \frac{x^3}{2.3} + \frac{x^5}{2.3.4.5} + \frac{3x^7}{2.4.5.6.7} + \frac{15x^9}{2.4.6.7.8.9} + \&c. + \frac{C\,x^2}{2} + C'x + C'';$$

and it is evident that each successive integration introduces a new arbitrary constant; consequently, the number of arbitrary constants must equal the number of successive integrations of the proposed differential.

In like manner, if we have

$$\int^3 x dx^3 = \int dx \int dx \int x dx,$$

by representing it by y, we have

$$y = \int dx \int dx \left(\frac{x^2}{2} + C\right) = \int dx \left(\frac{x^3}{6} + Cx + C'\right)$$

$$= \frac{x^4}{24} + \frac{Cx^2}{2} + C'x + C'';$$

which clearly represents a curve of the parabolic kind of the fourth order. Hence, if we have $\frac{d^n y}{dx^n} = X$, representing $\int X dx$ by X_1, $\int X_1 dx$ by X_2, and so on, we shall clearly have

$$\frac{d^{n-1}y}{dx^{n-1}} = \int X dx = X_1 + C,$$

$$\frac{d^{n-2}y}{dx^{n-2}} = \int X_1 dx + \int C dx = X_2 + C_1 x + C_2,$$

and so on, to any required extent.

2. It is manifest that these processes are applicable, whether the expressions to be integrated are of algebraic or transcendental forms. Thus we have

$$\int \cos x dx^3 = \int dx \int dx \int \cos x dx =$$

$$\int dx \int (\sin x + C)\, dx = -\sin x + \frac{Cx^2}{2} + C'x + C';$$

also

$$\int^3 e^x dx^3 = \int dx \int dx\, (e^x + C) =$$

$$\int dx\, (e^x + Cx + C') = e^x + \frac{Cx^2}{2} + C'x + C''.$$

3. Mr. Young, at p. 91 of his "Integral Calculus," gives

$$\int^n \mathrm{X}dx^n = \left(\int^n + dx^n\right) + \left(\int^{n-1} \mathrm{X}dx^{n-1}\right) x +$$

$$\left(\int^{n-2} \mathrm{X}dx^{n-2}\right) \frac{x^2}{1.2} + \ldots \left(\int \mathrm{X}dx\right) \frac{x^{n-1}}{1.2\ldots^{n-1}} +$$

$$(\mathrm{X}) \frac{x^n}{1.2\ldots\ldots n} + \left(\frac{d\mathrm{X}}{dx}\right) \frac{x^{n+1}}{1.2\ldots^{n+1}} +$$

$$\left(\frac{d^2\mathrm{X}}{dx^2}\right) \frac{x^{n+2}}{1.2\ldots\ldots^{n+2}} +, \&c.;$$

in which the development is made according to the ascending integral powers of x, by Maclaurin's theorem; $\left(\int^n \mathrm{X}dx^n\right)$ denoting the last of the arbitrary constants according to the preceding methods of development, $\left(\int^{n-1} \mathrm{X}dx^{n-1}\right)$ denoting the last constant but one, and so on until there are no arbitrary constants; noticing, that the terms within the parentheses stand for the values of the corresponding quantities, when x, in them, is put equal to naught.

EXAMPLES.

1. To develop $\int^4 \frac{dx^4}{\sqrt{(1+x^2)}}$ according to the ascending powers of x.

Since the binomial theorem gives

$$\frac{1}{\sqrt{1+x^2}} = 1 - \frac{1}{2}x^2 + \frac{1.3}{2.4}x^4 -, \&c.,$$

the development is

$$\int^4 \frac{dx^4}{\sqrt{1+x^2}} = C_4 + C_3 x + C_2 \frac{x^2}{2} + C_1 \frac{x^3}{2.3} + \frac{x^4}{2.3.4} -$$

$$\frac{x^6}{2.3.4.5.6} + \frac{1.3x^8}{2.3.4.5.6.7.8} - \&c. =$$

$$\frac{x^4}{2.3.4} - \frac{x^6}{2.3.4.5.6} + \&c. + C_1 \frac{x^3}{2.3} + C_2 \frac{x^2}{2} + C_3 x + C_4,$$

as given by Mr. Young.

2. To develop $\int^3 \sin x dx^3$ according to the ascending powers of x.

Here $\int^3 \sin x dx^3 = C_3 + C_2 x + C_1 \frac{x^2}{2} + \cos x$

$$= \cos x + C_1 \frac{x^2}{2} + C_2 x + C_3,$$

as in Young.

3. To develop $\frac{d^4 y}{dx^4}$ according to the ascending powers of x.

Here, we have

$$y = C_4 + C_3 x + C_2 \frac{x^2}{2} + C_1 \frac{x^3}{2.3};$$

which, in the language of curves, denotes a parabola of the third order.

4. To develop $\int^3 e^x dx^3$.

Here we have $\int^3 e^x dx^3 = C_3 + C_2 x + C_1 \frac{x^2}{2} + e^x$.

(13.) We now propose to show the use of arbitrary constants in finding definite integrals by series or otherwise.

According to what is shown at p. 265, the notation

$$\int_0^1 \frac{dx}{\sqrt{(1-x^2)}} = \frac{\pi}{2}$$

signifies that the integral being taken from $x = 0$ to $x = 1$, gives $\frac{\pi}{2}$ = one-fourth of the circumference of a circle whose radius = 1, for the result or value of the integral contained between the preceding limits; and a like notation is to be used in all analogous cases of definite integrals.

If we take the integrals indicated in example 7, at p. 386, from $x = 0$ to $x = 1$, when m stands for an odd or even positive integer; then, for m odd, we have the results

$$\int_0^1 \frac{xdx}{\sqrt{(1-x^2)}} = 1, \int_0^1 \frac{x^3dx}{\sqrt{(1-x^2)}} = \frac{2}{3}, \int_0^1 \frac{x^5dx}{\sqrt{(1-x^2)}} = \frac{2.4}{3.5},$$

$$\int_0^1 \frac{x^7dx}{\sqrt{(1-x^2)}} = \frac{2.4.6}{3.5.7} \ldots\ldots \int_0^1 \frac{x^{2n+1}dx}{\sqrt{(1-x^2)}} = \frac{2.4.6\ldots 2n}{3.5.7\ldots(2n+1)};$$

by using $2n + 1$ for m, and by proceeding in like manner for m even, we shall have

$$\int_0^1 \frac{dx}{\sqrt{(1-x^2)}} = \frac{\pi}{2}, \int_0^1 \frac{x^2dx}{\sqrt{(1-x^2)}} = \frac{1}{2}\cdot\frac{\pi}{2},$$

$$\int_0^1 \frac{x^4dx}{\sqrt{1-x^2}} = \frac{1.3}{2.4}\cdot\frac{\pi}{2}, \int_0^1 \frac{x^6dx}{\sqrt{(1-x^2)}} = \frac{1.3.5}{2.4.6}\cdot\frac{\pi}{2},$$

$$\ldots\ldots \int_0^1 \frac{x^{2n}dx}{\sqrt{(1-x^2)}} = \frac{1.3.5.7\ldots(2n-1)}{2.4.6.8\ldots 2n}\cdot\frac{\pi}{2},$$

by using $2n$ for m.

It is easy to perceive, from a comparison of the preceding values, that if n is large we shall have

$$\int_0^1 \frac{x^{2n}dx}{\sqrt{(1-x^2)}} = \int_0^1 \frac{x^{2n+1}dx}{\sqrt{(1-x^2)}} \text{ nearly},$$

or $$\frac{1.3.5.7\ldots\ldots(2n-1)}{2.4.6.8\ldots\ldots 2n}\cdot\frac{\pi}{2} = \frac{2.4.6\ldots\ldots 2n}{3.5.7\ldots\ldots(2n+1)} \text{ nearly},$$

or we shall have $$\frac{\pi}{2} = \frac{2.2.4.4.6.6\ldots\ldots 2n.2n}{1.3.3.5.5.7\ldots(2n-1).(2n+1)} \text{ nearly};$$

and by supposing n to be unlimitedly great, or infinitely great, we must evidently have

$$\frac{\pi}{2} = \frac{2.2.4.4\ldots\ldots 2n.2n}{1.3.3.5.5\ldots\ldots(2n-1).(2n+1)} \text{ exactly,}$$

for the length of the quadrantal arc of a circle whose radius $= 1$: where it may be noticed that this expression seems to have been first discovered by Dr. Wallis. (See Young, pp. 97 and 98, and Lacroix, vol. iii., p. 415.)

Remarks.—Mr. Young, although he has with reason objected to the manner in which the formula of Wallis is frequently written by English authors, yet, at p. 97 of his work, he has written

$$\frac{2.2.4.4.6.6.8.8}{1.3.3.5.5.7.7.9.9} \text{ for } \frac{\pi}{2}, \text{ instead of } \frac{2.2.4.4.6.6.8.8}{1.3.3.5.5.7.7.9},$$

which is its proper form when the numerator and denominator each consists of eight factors; noticing, that the numerator and denominator of the fractional forms of $\frac{\pi}{2}$ must each consist of the same number of factors as the preceding forms.

If we write the successive approximate values of $\frac{\pi}{2}$, after the factors common to their numerators and denominators are rejected, we shall have

$$\frac{2.2}{1.3} = \frac{4}{3} = 1.3, \ \frac{2.2.4.4}{1.3.3.5} = \frac{64}{45} = 1 + \frac{19}{45}, \ \frac{2.2.4.4.2.2}{1.1.1.5.5.7} = 1 + \frac{81}{175},$$

$$\frac{2.2.4.4.2.2.8.8}{1.1.1.5.5.7.7.9} = 1.48+, \ \frac{2.2.4.4.2.2.8.8.2.2}{1.1.1.1.1.7.7.9.9.11} = 1.50+,$$

and so on. From these results it is clear that the successive terms approximate very slowly to the length of the quadrantal arc, the last result being correct only to one place of decimals.

For another example, we will show how to find the development of the definite integral $\int_0^1 \frac{dx}{\sqrt{(1-x^4)}}$.

Because $1 - x^4 = (1 - x^2)(1 + x^2)$, we shall clearly have

$$\int_0^1 \frac{dx}{\sqrt{(1-x^4)}} = \int_0^1 \frac{dx}{\sqrt{(1-x^2)}} \times (1 + x^2)^{-\frac{1}{2}}$$

$$= \int_0^1 \frac{dx}{\sqrt{(1-x^2)}} \left(1 - \frac{x^2}{2} + \frac{13x^4}{2.4} - \frac{1.3.5}{2.4.6} x^6 + \&c.\right)$$

$$= \int_0^1 \frac{dx}{\sqrt{(1-x^2)}} - \frac{1}{2}\int_0^1 \frac{x^2dx}{\sqrt{(1-x^2)}} + \frac{1.3}{2.3}\int_0^1 \frac{x^4dx}{\sqrt{(1-x^2)}} -$$

$$\frac{1.3.5}{2.4.6}\int \frac{x^6dx}{\sqrt{(1-x^2)}} +, \&c.$$

Because, from what is shown in the preceding examples we have

$$\int_0^1 \frac{dx}{\sqrt{(1-x^2)}} = \frac{\pi}{2}, \int_0^1 \frac{x^2dx}{\sqrt{(1-x^2)}} = \frac{1}{2} \cdot \frac{\pi}{2},$$

$$\int_0^1 \frac{x^4dx}{\sqrt{(1-x^2)}} = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{\pi}{2},$$

and so on, we get by substitution

$$\int_0^1 \frac{dx}{\sqrt{(1-x^4)}} = \left\{1 - \left(\frac{1}{2}\right)^2 + \left(\frac{1.3}{2.4}\right)^2 \quad \left(\frac{1.3.5}{2.4.6}\right)^2 + \&c.\right\} \frac{\pi}{2}.$$

For a final example we will find the integral $\int_0^1 \frac{dx}{\sqrt{x(1-x^2)}}$.

By putting $y = 2\sqrt{x}$ we get $dy = \frac{dx}{\sqrt{x}}$, $x^2 = \left(\frac{y}{2}\right)^4$; consequently,

$$\int_0^1 \frac{dx}{\sqrt{x(1-x^2)}} = 2\int_0^1 \frac{\frac{dy}{2}}{\sqrt{\left\{1-\left(\frac{y}{2}\right)^4\right\}}} = 2\int_0^1 \frac{dz}{\sqrt{(1-z^4)}}$$

by putting $\frac{y}{2} = z$.

Hence, from the preceding example, we shall have

$$\int_0^1 \frac{dx}{\sqrt{x(1-x^2)}} = 2\int_0^1 \frac{dz}{\sqrt{(1-z^4)}} = 2\left\{1-\left(\frac{1}{2}\right)^2 + \&c.\right\}\frac{\pi}{2}$$

$$= \left\{1-\left(\frac{1}{2}\right)^2 + \left(\frac{1.3}{2.4}\right) - \left(\frac{1.3.5}{2.4.6}\right)^2 + \&c.\right\}\pi;$$

which is twice the integral found in the preceding example.

(14.) We will terminate this section by showing how to sum series, or to find their generating functions, by means of the preceding principles.

The processes here proposed seem to depend, for the most part, on transforming, by means of the integral or differential calculus, the proposed series into a new series, or in finding a new series, such that its sum or generating function can be found, so that the proposed series may be supposed to have been derived from it.

EXAMPLES.

1. To find the sum of the series

$$s = x + 2x^2 + 3x^3 + \ldots + nx^n.$$

Multiplying the members of this equation by $\frac{dx}{x}$ and taking the integrals of the products, we have

$$\int s\frac{dx}{x} = \int (dx + 2x dx + 3x^2 dx + \&c.)$$

$$= x + x^2 + x^3 + \ldots + x^n,$$

which is clearly a geometrical progression; whose sum, by the common rule, clearly equals $\frac{x - x^{n+1}}{1 - x}$, and thence we have $\int s \frac{dx}{x} = \frac{x - x^{n+1}}{1 - x}$. To find s, the sum of the proposed series, from this, we must remove the sign of integration $\int$, by taking the differentials of its members, which give

$$s \frac{dx}{x} = \frac{dx - (n + 1) x^n dx + nx^{n+1} dx}{(1 - x)^2},$$

or, by a simple reduction, we have

$$s = \frac{x - (n + 1) x^{n+1} + nx^{n+2}}{(1 - x)^2}.$$

2. To find the generating function of the same series continued indefinitely.

It is manifest from development, that we shall here have $s = \frac{x}{(1 - x)^2}$; which is clearly the same as to suppose the definite parts, or those that depend on n, in the preceding sum, to destroy each other, and to put $s = \frac{x}{(1 - x)^2}$

3. To find the generating function of the series

$$1 + x + \frac{x^2}{2} + \frac{x^3}{2.3} + \frac{x^4}{2.3.4} + \text{\&c.}$$

Denote the sum by y, and we shall have

$$y = 1 + x + \frac{x^2}{2} + \frac{x^3}{2.3} +, \text{\&c.};$$

whose differential coefficients give

$$\frac{dy}{dx} = 1 + x + \frac{x^2}{2} + \frac{x^3}{2.3} + \text{\&c.} = y;$$

consequently, we shall thence get $\frac{dy}{y} = dx$. By taking the integrals of the members of this equation, we have $\log y = x$, which needs no correction, supposing it to commence with x, since $x = 0$ gives $y = 1$, whose $\log = 0$; consequently, putting e for the hyperbolic base, we shall, by the nature of logarithms, have $y = e^x$ and thence

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{2.3} +, \&c.,$$

as required.

4. To find the sum of

$$s = \frac{x^{n+1}}{n+1} + \frac{x^{n+3}}{2.3(n+3)} + \frac{x^{n+5}}{2.3.5(n+5)} +, \&c.$$

From (b''), at p. 51, we have

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{2.3} + \frac{x^4}{2.3.4.5} +, \&c.;$$

and putting $-x$ for x in this, we also have

$$e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{2.3} +, \&c.,$$

whose half difference gives

$$\frac{e^x}{2} - \frac{e^{-x}}{2} = x + \frac{x^3}{2.3} + \frac{x^5}{2.3.4.5} +, \&c.;$$

and multiplying the members of this by $x^{n-1}\,dx$, we have

$$\frac{1}{2}\int e^x x^{n-1}\,dx - \frac{1}{2}\int e^{-x} x^{n-1}\,dx =$$

$$\frac{x^{n+1}}{n+1} + \frac{x^{n+3}}{2.3(n+3)} + \frac{x^{n+5}}{2.3.4.5(n+5)} +, \&c.,$$

since the method of integration, explained at pp. 391 to 393, reduces the equation to

$$s = \frac{1}{2}\int e^x x^{n-1} dx \frac{1}{2}\int e^x x^{n-1} dx$$

$$= \frac{1}{2} e^x [x^{n-1} - (n-1)x^{n-2} + (n-1)(n-2)x^{n-3} - \&c.] +$$

$$\frac{1}{2} e^{-x} [x^{n-1} + (n-1)x^{n-2} + (n-1)(n-2)x^{n-3} + \&c.];$$

consequently, from equating the values of s, we shall have

$$\frac{1}{2} e^x [x^{n-1} - (n-1)x^{n-2} + (n-1)(n-2)x^{n-3} - \&c.] +$$

$$\frac{1}{2} e^{-x} [x^{n-1} + (n-1)x^{n-2} + (n-1)(n-2)x^{n-3} + \&c.] =$$

$$\frac{x^{n+1}}{n+1} + \frac{x^{n+3}}{2.3(n+3)} + \frac{x^{n+5}}{2.3.4.5(n+5)} +, \&c.,$$

which needs no correction, supposing its members to commence with x. If we put $n = 2$ and $x = 1$ in this equation, we have $e^{-1} = \frac{1}{e} = \frac{1}{3} + \frac{1}{2.3.5} + \frac{1}{2.3.4.5.7} +, \&c.;$

which is the same result that Mr. Young has found at p. 100 of his "Integral Calculus," from which the example has been taken.

5. To find the generating function of the series whose general term may be expressed by the term

$$\frac{1}{(p+qn)(r+sn)\ \&c.},$$

in which n stands for the number or place of the term in the series.

Because $\frac{x^{\frac{p}{q}}}{1 \mp x} = x^{\frac{p}{q}} \pm x^{\frac{p}{q}+1} + x^{\frac{p}{q}+2} \pm, \&c.,$

if we multiply the members of this equation by dx and take the integrals of the products, we shall have

$$\frac{1}{q}\int \frac{x^{\frac{p}{q}}}{1 \mp x}\,dx = \frac{x^{\frac{p}{q}+1}}{p+q} \pm \frac{x^{\frac{p}{q}+2}}{p+2q} + \frac{x^{\frac{p}{q}+3}}{p+3q} \pm, \text{\&c.};$$

consequently, if X_1 represents the sought function, we shall have

$$X_1 = \frac{1}{q}\int \frac{x^{\frac{p}{q}}}{1 \mp x}\,dx = \frac{x^{\frac{p}{q}+1}}{p+q} \pm \frac{x^{\frac{p}{q}+2}}{p+2q} + \frac{x^{\frac{p}{q}+3}}{p+3q} \pm, \text{\&c.},$$

which, since its right member vanishes when $x = 0$, we shall suppose its members to be so taken that they both commence with x, and extend to $x = 1$. Thus, if $p = 0$, we have

$$X_1 = -\frac{1}{q}\int \frac{-\,dx}{1-x} = -\frac{1}{q}\log(1-x) = \frac{x}{q} + \frac{x^2}{2q} + \frac{x^3}{3q} +, \text{\&c.},$$

when we take $1 - x$ for $1 \mp x$; and if we put $x = 0$ in the members of this they both vanish, while if we put $x = 1$ in the members they reduce to

$$\frac{1}{q}\log 0 = \text{infinity} = \frac{1}{q}\left(1 + \frac{1}{2} + \frac{1}{3} + \text{\&c.}\right),$$

a well-known result. Again, if we take $1 + x$ for $1 \mp x$ we shall, as before, by putting $p = 0$, get

$$\frac{1}{q}\log 2 = \frac{1}{q}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \text{\&c.}\right)$$

or $$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} +, \text{\&c.}$$

Resuming,

$$X_1 = \frac{1}{q}\int \frac{x^{\frac{p}{q}}\,dx}{1 \mp x} = \frac{x^{\frac{p}{q}+1}}{p+q} \pm \frac{x^{\frac{p+2}{q}}}{p+2q} + \frac{x^{\frac{p}{q}+3}}{p+3q} \pm, \text{\&c.};$$

then by multiplying its members by $x^{\frac{r}{s}-\frac{p}{q}-1}\,dx$, and integrating by putting $X_2 = \frac{1}{s}\int X_1\, x^{\frac{r}{s}-\frac{p}{q}-1}\,dx$, we get

$$X_2 = \frac{x^{\frac{r}{s}+1}}{(p+q)(r+s)} \pm \frac{x^{\frac{r}{s}+2}}{(p+2q)(r+2s)} +, \text{ \&c.}$$

Multiplying the members of this by $x^{\frac{t}{u}-\frac{r}{s}-1}\,dx$,

and putting $$X_3 = \frac{1}{u}\int X_2\, x^{\frac{t}{u}-\frac{r}{s}-1}\,dx$$

and integrating, we shall have

$$X_3 = \frac{x^{\frac{t}{u}+1}}{(p+q)(r+s)(t+u)} \pm \frac{\frac{t}{u}+2}{(p+2q)(r+2s)(t+2u)} +, \text{ \&c.};$$

and so on, to any required extent.

Since $$X_2 = \frac{1}{s}\int X_1\, x^{\frac{r}{s}-\frac{p}{q}-1}\,dx,$$

when there are but two factors in the denominator, we get, from substituting the value of X_1,

$$\frac{1}{qs}\int x^{\frac{r}{s}-\frac{p}{q}-1}\,dx \int \frac{x^{\frac{p}{q}}}{1 \mp x}\,dx;$$

which, integrated by parts, gives

$$\frac{x^{\frac{r}{s}-\frac{p}{q}}}{qs\left(\frac{r}{s}-\frac{p}{q}\right)}\int \frac{x^{\frac{p}{q}}}{1 \mp x}\,dx - \frac{1}{qs\left(\frac{r}{s}-\frac{p}{q}+1\right)}\int \frac{x^{\frac{r}{s}}}{1 \mp x}\,dx$$

for its value. Supposing the integral to be taken from $x = 0$ to $x = 1$, we shall have

$$\frac{1}{s}\int X_1\, x^{\frac{r}{s}-\frac{p}{q}-1}\,dx =$$

$$\frac{x^{\frac{r}{s}-\frac{p}{q}}}{qs\left(\frac{r}{s}-\frac{p}{q}\right)}\int_0^1 \frac{x^{\frac{p}{q}}\,dx}{1 \mp x} - \frac{1}{qs\left(\frac{r}{s}-\frac{p}{q}\right)}\int_0^1 \frac{x^{\frac{r}{s}}}{1 \mp x}\,dx \ldots. (a').$$

An example or two may serve for illustration.

1st. To find the generating function of the infinite series

$$X_2 = \frac{1}{1.4} - \frac{1}{2.5} + \frac{1}{3.6} - \frac{1}{4.7} +, \&c.$$

The series is clearly satisfied by putting $p = 0$, $q = 1$, and $r = 3$, $s = 1$, and thence (a') becomes

$$X_2 = \frac{1}{3}\int_0^1 \frac{dx}{1+x} - \frac{1}{3}\int_0^1 \frac{x^3}{1+x}\,dx = \frac{2}{3}\log 2 - \frac{5}{18} + \text{const.}$$

2d. To find the generating function of the infinite series

$$X_2 = \frac{1}{1.4} - \frac{1}{3.6} + \frac{1}{5.8} -, \&c.$$

Since $p = -1$, $q = 2$, and $r = 2$, $s = 2$, (a') gives

$$X_2 = \frac{1}{4}\int_0^1 \frac{x^{-\frac{1}{2}}\,dx}{1+x} - \frac{1}{4}\int_0^1 \frac{x^{\frac{1}{2}}\,dx}{1+x}.$$

To find the first of these integrals, we put $y = x^{\frac{1}{2}}$, and thence get

$$dy = \frac{1}{2}x^{-\frac{1}{2}}dx - x = y^2,$$

and thence the first integral becomes

$$\frac{1}{4}\int_0^1 \frac{2dy}{1+y^2} = \frac{\pi}{4}, \quad \text{and} \quad -\frac{1}{4}\int_0^1 \frac{x^{\frac{1}{2}}\,dx}{1+x} = -\frac{1}{4}\log 2;$$

consequently, $X_2 = \frac{\pi}{4} - \frac{1}{4}\log 2$ equals the generating function of the series, or $X_2 = \frac{\pi}{4} - \log\sqrt{2}$, as required.

6. We now propose to show how to find generating functions that may be reduced to the form

$$s = \frac{a}{p+q} \pm \frac{b}{p+2q} + \frac{c}{p+3q} \pm, \&c.$$

We shall assume the series

$$s = \frac{ax^{\frac{p}{q}+1}}{p+q} \pm \frac{bx^{\frac{p}{q}+2}}{p+2q} + \frac{cx^{\frac{p}{q}+3}}{p+3q} \pm, \text{\&c.},$$

which vanishes when $x = 0$, and becomes the proposed series when $x = 1$. Hence, by taking the differential of the members of the assumed series, we shall have

$$qds = ax^{\frac{p}{q}}dx \pm bx^{\frac{p}{q}+1}\,dx + cx^{\frac{p}{q}+2} \pm dx, \text{\&c.},$$

whose integral being taken from $x = 0$ to $x = 1$, gives

$$qs = \int_0^1 (ax^{\frac{p}{q}} \pm bx^{\frac{p}{q}+1} + cx^{\frac{p}{q}+2} \pm \text{\&c.})\,dx \; . \; . \; . \; . \; . \; (b),$$

which will clearly give the value of the proposed series when the integral can be found.

Thus, to find the limiting function of

$$s = \frac{1}{3} - \frac{2}{4} + \frac{3}{5} - \frac{4}{6} +, \text{\&c.},$$

in which 1, 2, 3, &c., represent the letters a, b, c, &c., while 3, 4, 5, &c., are represented by $p + q$, $p + 2q$, $p + 3q$, &c., it is manifest that we must have $p = 2$ and $q = 1$, since $-$ is used for $\pm$ in the example; and that $x^{\frac{p}{q}}$ may be moved without the parenthesis, we shall have, from (b),

$$qs = \int_0^1 (a - bx + cx^2 - dx^3 + \text{\&c.})\,dx^{\frac{p}{q}}\,dx,$$

which the substitution of the preceding values of a, b, c, &c., reduces to $\qquad s = \int_0^1 \frac{x^2}{(1+x)^2}\,dx.$

By taking the general integral, we have

$$s = x - 2\log(1+x) - \frac{1}{x+1} + C,$$

which, being taken from $x = 0$ to $x = 1$, gives $s = 4 - 2\log 2$ nearly, for the generating function of the proposed series.

7. We now propose to show how to find the generating function of a series of the form

$$s = \frac{1}{(p+q)m} \pm \frac{1}{(p+2q)m^2} + \frac{1}{(p+3q)m^3} \pm, \text{\&c.}$$

Assuming

$$s = \frac{x^{\frac{p}{q}+1}}{(p+q)m} \pm \frac{x^{\frac{p}{q}+2}}{(p+2q)m^2} + \frac{x^{\frac{p}{q}+3}}{(p+3q)m^3} \pm, \text{\&c.},$$

then, as before, we shall have

$$qds = \frac{x^{\frac{p}{q}}}{m}dx \pm \frac{x^{\frac{p}{q}+1}}{m^2}dx + \frac{x^{\frac{p}{q}+2}}{m^3}dx \pm \text{\&c.}$$

$$= \left(\frac{1}{m} \pm \frac{x}{m^2} + \frac{x^2}{m^3} \pm \text{\&c.}\right) x^{\frac{p}{q}}dx = \frac{x^{\frac{p}{q}}}{m \mp x}dx,$$

whose general integral is

$$qs = \int \frac{x^{\frac{p}{q}}dx}{m \mp x};$$

consequently, we shall have

$$s = \frac{1}{q}\int_0 \frac{x^{\frac{p}{q}}}{m \mp x}dx \quad \ldots\ldots\ldots\ldots \quad (c).$$

Thus, to find the generating function of

$$s = \frac{1}{2.2} - \frac{1}{3.4} + \frac{1}{4.8} -, \text{\&c.}$$

Here, p and q are each 1, and m, m^2, m^3, &c., are 2, 4, and 8, &c.; consequently, since we must clearly use $+$ for $\mp$, we have

$$qs = \int \frac{x}{2+x}\,dx, \quad \text{and thence} \quad qs = x - 2\log(2+x) + C$$

is the general integral, which, taken between the preceding limits, gives

$$s = \frac{1}{q}\int_0^1 \frac{x}{2+x}\,dx = 1 + 2\sqrt{2} - 2\sqrt{3},$$

as required.

8. To illustrate what is sometimes called Lorgna's method of series, we will apply it to one or two examples.

1st. To find the generating function of the infinite series

$$s = \frac{1}{1.2} - \frac{1}{2.3} + \frac{1}{3.4} -, \&c.$$

Because we have

$$X_1 = \int \frac{dx}{1+x} = x - \frac{x^2}{2} + \frac{x^3}{3} -, \&c.,$$

by multiplying the members of this by dx and integrating, we have

$$X_1\,dx = X_2 = \int dx \int \frac{dx}{1+x} = \frac{x^2}{1.2} - \frac{x^3}{2.3} + \frac{x^4}{3.4} -, \&c.$$

By taking the integral by parts, we have

$$\int dx \int \frac{dx}{1+x} = x\log(1+x) - \int \frac{x}{1+x}\,dx$$
$$= (x+1)\log(1+x) - x;$$

consequently, we have

$$(x+1)\log(1+x) - x = \frac{x^2}{1.2} - \frac{x^3}{2.3} + \frac{x^4}{3.4} -, \&c.,$$

which needs no correction, supposing the integrals to begin with x; and thence, by putting 1 for x, we have

$$2\log 2 - 1 = \frac{1}{1.2} - \frac{1}{2.3} + \frac{1}{3.4} -, \&c.,$$

for the generating function of the given series.

2d. To find the generating function of the infinite series

$$s = \frac{1}{1.2.4} - \frac{1}{2.3.5} + \frac{1}{3.4.6} -, \text{\&c.}$$

Proceeding, as in the preceding example, we have

$$s = \int x dx \int dx \int \frac{dx}{1+x} = \frac{x^4}{1.2.4} - \frac{x^5}{2.3.5} + \frac{x^6}{3.4.6} -, \text{\&c.}$$

From the last example we have

$$\int dx \int \frac{dx}{1+x} = (x+1)\log(1+x) - x,$$

and thence we shall have

$$\int x dx \int dx \int \frac{dx}{1+x} = \int x(x+1)\, dx \log(1+x) - \int x^2\, dx;$$

which, integrated by parts, gives the integral

$$s = \left(\frac{x^3}{3} + \frac{x^2}{2}\right)\log(1+x) - \frac{x^3}{3} - \int\left(\frac{x^3}{3} + \frac{x^2}{2}\right)\frac{dx}{1+x},$$

and thence the integral reduces to

$$\left(\frac{x^3}{3} + \frac{x^2}{2} - \frac{1}{6}\right)\log(1+x) - \frac{4x^3}{9} - \frac{x^2}{12} + \frac{x}{6}$$

$$= \frac{x^4}{1.2.4} - \frac{x^5}{2.3.5} + \frac{x^6}{3.4.6} -, \text{\&c.},$$

which needs no correction, supposing the integral to commence with x. If we put $x = 1$, we have

$$s = \frac{2}{3}\log 2 - \frac{13}{36} = \frac{1}{1.2.4} - \frac{1}{2.3.5} + \frac{1}{3.4.6} -, \text{\&c.},$$

for the generating function as required.

3d. To find the generating function of the infinite series

$$\frac{1}{2^2.4^2} + \frac{1^2}{2^2.4^2.6^2} + \frac{1^2.3^2}{2^2.4^2.6^2.8^2} + \frac{1^2.3^2.5^2}{2^2.4^2.6^2.8^2.10^2} +, \text{\&c.}$$

From what is done at p. 387, we easily obtain the formula

$$\int_0^1 \frac{x^{2n}\,dx}{\sqrt{(1-x^2)}} = \frac{1.3.5\,\ldots\,(2n-1)}{2.4.6\,\ldots\ldots\,2n}\,\frac{\pi}{2},$$

in which n is an arbitrary positive integer, which, by putting 2, 3, 4, &c., successively, for n, enables us, by representing the generating function by s, to write the form

$$\frac{\pi}{2}s = \frac{1}{2.3.4}\int\frac{x^4\,dx}{\sqrt{(1-x^2)}} + \frac{1}{2.3.4.5.6}\int\frac{x^6\,dx}{\sqrt{(1-x^2)}} +$$

$$\frac{1^2.3^2}{2.3.4.5.6.7.8}\int\frac{x^8\,dx}{\sqrt{(1-x^2)}} +$$

$$\frac{3^2.5^2}{2.3.4.5.6.7.8.9.10}\int\frac{x^{10}}{\sqrt{(1-x^2)}} +, \text{\&c.},$$

whose integrals, being taken from $x = 0$ to $x = 1$, will equal the proposed series multiplied by $\frac{\pi}{2}$.

By taking the differentials of the members of this equation, it is immediately reduced to the form

$$\frac{\pi}{2}\frac{ds}{dx}\sqrt{(1-x^2)} = \frac{x^4}{2.3.4} + \frac{x^6}{2.3.4.5.6} +$$

$$\frac{1^2.3^2\,x^8}{2.3.4.5.6.7.8} + \frac{3^2.5^2\,x^{10}}{2.3.4.5.6.7.8.9.10} +, \text{\&c.};$$

and by differentiating the numbers of this equation three times successively, regarding dx as invariable, we have

$$\frac{\pi}{2}\times d^3\left(\frac{ds}{dx}\sqrt{(1-x^2)}\right) =$$

$$dx^3\left(x + \frac{x^3}{2.3} + \frac{1^2.3^2\,x^6}{2.3.4.5} + \frac{3^2.5^2\,x^7}{2.3.4.5.6.7} + \text{\&c.}\right).$$

Since the right member of this equation (between the parentheses) is the same as $\sin^{-1} x$, we hence get the equation

$$\frac{\pi}{2} \times d^3 \left(\frac{ds}{dx} \sqrt{(1 - x^2)}\right) = \sin^{-1} x dx^3,$$

whose integrals give

$$\frac{\pi}{2} \int^3 d^3 \left(\frac{ds}{dx} \sqrt{(1 - x^2)}\right) = \int^2 dx^2 \int \sin^{-1} x dx.$$

By taking the integral of the members of this equation, we have

$$\frac{\pi}{2} \int^2 d^2 \left(\frac{ds}{dx} \sqrt{(1 - x^2)}\right) = \int^2 (-\cos^{-1} x + 1)\, dx^2;$$

and taking the integral of this, we also easily get

$$\frac{\pi}{2} \int d.\left(\frac{ds}{dx} \sqrt{(1 - x^2)}\right) = \int (-\sin^{-1} x + x)\, dx;$$

whose integral again taken becomes

$$\frac{\pi}{2} \frac{ds}{dx} \sqrt{(1 - x^2)} = -\cos^{-1} x + \frac{x^2}{2}$$

or $$\int ds = \frac{2}{\pi} \int \frac{x^2\, dx}{2\sqrt{(1 - x^2)}} - \frac{2}{\pi} \int \frac{\cos x dx}{\sqrt{(1 - x^2)}},$$

and so on.

REMARKS.—The substance of the last six pages has been taken from Young's "Integral Calculus," from pp. 99 to 111 inclusive. Mr. Y. shows, at p. 108, in a manner very analogous to that used by us in the solution of our last example, that the generating function of the series

$$\frac{3^2}{4^2.6} + \frac{3^2.5^2}{4^2.6^2.8} + \frac{3^2.5^2.7^2}{4^2.6^2.8^2.10} +, \&c., \text{ equals } \frac{8}{\pi} - 2\frac{1}{4}.$$

SECTION V.

INTEGRATION OF DIFFERENTIAL EXPRESSIONS WHICH CONTAIN TWO OR MORE VARIABLES.

(1.) A DIFFERENTIAL of a function of two or more variables which is derived from the function by taking its differential, supposing the variables all to change, is said to be *complete* or *exact*; while, if the differential is taken on the supposition that the variables do not all change their values, it is said to be *incomplete*, *inexact*, or *partial*.

Thus,

$$ydx + xdy = d\overline{yx}, \quad \frac{dy}{x} - \frac{ydx}{x^2} = \frac{xdy - ydx}{x^2} = d\,\frac{y}{x},$$

are complete or exact differentials, while ydx, $xdy - ydx$, are incomplete or inexact differentials, provided there is no assigned relationship between x and y; other examples of exact differentials will be obtained by reversing the examples at pp. 7 to 12.

(2.) It is easy to perceive that if $\mathrm{M}dx + \mathrm{N}dy$ is an exact differential of two variables x and y, that its integral may be found by the following

RULE.

1. Take the integral $\int \mathrm{M}dx$ on the supposition that y is constant or invariable, and add to the result the integral of all the terms in $\mathrm{N}dy$ which are independant of x or do not

contain x; then the result, increased by an arbitrary constant, will be the complete or exact integral.

2. Or we may take the integral $\int N dy$, on the supposition of the constancy of x, and increase the result by the integral of that part of $M dx$ which is independent of y, and an arbitrary constant for the same integral as before.

REMARKS.—It clearly results from the rule, that when $M dx + N dy$ is an exact or complete differential of a function (M and N being functions of x and y) we must have $\frac{dM}{dy} = \frac{dN}{dx}$; which is called *Euler's Criterion* or *Condition of Integrability* of the differential $M dx + N dy$ (see p. 22) Hence, since $\int M dx = \int N dy$, we have

$$\frac{d \int M dx}{dy} = N;$$

and from $\frac{dN}{dx} = \frac{dM}{dy}$, we have $dN = \frac{dM}{dy} dx$, which gives $N = \int \frac{dM}{dy} dx$; consequently, we must have

$$\frac{d \int M dx}{dy} = \int \frac{dM}{dy} dx,$$

which is agreeable to Leibnitz's rule for differentiating under the sign $\int$; noticing, that the right member of this equation is independent of the first integral, or that with respect to x.

(3.) To illustrate the rule, take the following

EXAMPLES.

1. To find the integral of $(6xy - y^2)\,dx + (3x^2 - 2xy)\,dy$.

Since x enters into every term of the coefficient of dy, it is clear, if the proposed differential is exact, it will be sufficient to find the integral $\int(6xy - y^2)\,dx$, supposing y constant; consequently, $3x^2y - y^2x + C$ must be the integral, which is evidently true, since it equals the integral

$$\int(3x^2 - 2xy)\,dy,$$

regarding x as constant.

REMARK.—Because, in this example, M and N are represented by $6xy - y^2$ and $3x^2 - 2xy$, which give $\frac{dM}{dy} = 6x - 2y$ and $\frac{dN}{dx} = 6x - 2y$, the criterion of integrability, $\frac{dM}{dy} = \frac{dN}{dx}$, is satisfied.

2. To find the integral of $(3x^2 + 2axy)\,dx + (ax^2 + 3y^2)\,dy$.

Here $$\int(3x^2 + 2axy)\,dx = x^3 + ax^2y,$$

to which adding the integral of $3y^2dy$, the part of $(ax^2 + 3y^2)\,dy$ which is independent of x, and we have $x^3 + ax^2y + y^3 + C$, after adding the constant C, for the exact integral.

The same integral is also found from the integral

$$\int(ax^2 + 3y^2)\,dy = ax^2y + y^3,$$

by adding the integral $3\int x^2\,dx = x^3 + C$, the integral of the part of $(3x^2 + 2axy)\,dy$ which is independent of y, to it.

The criterion is also satisfied.

3. To find the integral of $\frac{ydx - xdy}{x^2 + y^2} = \frac{ydx}{y^2 + x^2} - \frac{xdy}{y^2 + x^2}$.

Here the integral

$$\int \frac{ydx}{y^2 + x^2} = \int \frac{\frac{dx}{y}}{1 + \frac{x^2}{y^2}} = \tan^{-1}\frac{x}{y} + \text{C};$$

and the integral

$$\int - \frac{xdy}{y^2 + x^2} = \int \frac{-x\frac{dy}{y^2}}{1 + \frac{x^2}{y^2}} = \tan^{-1}\frac{x}{y} + \text{C},$$

the same as before.

4. To find the integral of $\frac{x^2dx}{\sqrt{(a^2 + x^2)}} + ydy$.

Here $\text{M} = \frac{x^2}{\sqrt{(a^2 + x^2)}}$ and $\text{N} = y$, which, since they do not contain y and x, give $\frac{d\text{M}}{dy} = 0$ and $\frac{d\text{N}}{dx} = 0$, which, being naught, may be regarded as satisfying the criterion of integrability.

Hence, the proposed differential may be regarded as having an exact integral, which is also evident from principles heretofore given, since each term of the proposed differential is clearly the function of a single variable. Indeed, the integral

$$\int \frac{x^2dx}{\sqrt{(a^2 + x^2)}} = \frac{x\sqrt{(a^2 + x^2)}}{2} - \frac{a^2}{2}\int \frac{dx}{\sqrt{(a^2 + x^2)}}$$

$$= \frac{x\sqrt{(a^2 + x^2)}}{2} - \frac{a^2}{2}\log\left(x + \sqrt{x^2 + a^2}\right),$$

and since $\int ydy = \frac{y^2}{2}$, the integral of the proposed differential is of course found, after the addition of the arbitrary constant.

5. To find the integral of

$$(3ax^2 + 2bxy)\,dx + bx^2dy.$$

Here we have

$$\int \mathrm{M}dx = \int (3ax^2 + 2bxy)\,dx = ax^3 + bx^2y + \mathrm{Y},$$

in which Y stands for the arbitrary constant in the integration with regard to x while it may be a function of y, since y has been supposed to be constant in the integration with reference to x.

To determine Y, we take the differential of the preceding equation, regarding x as constant, and thence get

$$\mathrm{N} = \frac{d\int \mathrm{M}dx}{dy} + \frac{d\mathrm{Y}}{dy} \quad \text{or} \quad \mathrm{Y} = \int \left(\mathrm{N} - \frac{d\int \mathrm{M}dx}{dy}\right) dy;$$

consequently, since

$$\mathrm{N} = bx^2 \quad \text{and} \quad \frac{d\int \mathrm{M}dx}{dy} = bx^2,$$

we have
$$\mathrm{N} - \frac{d\int \mathrm{M}dx}{dy} = 0.$$

Hence, the sought integral is reduced to

$$\int \mathrm{M}dx + \int \left(\mathrm{N} - \frac{d\int \mathrm{M}dx}{dy}\right) dy = ax^3 + bx^2y + \mathrm{C};$$

which might also have been expressed by the form

$$\int \mathrm{N}dy + \int \left(\mathrm{M} - \frac{d\int \mathrm{N}dy}{dx}\right) dx = bx^2y + ax^3 + \mathrm{C},$$

the same as before.

REMARK.—We have performed this solution according to the common methods, in order to show that they are substantially the same as our rule.

(4.) It is easy to perceive that our rule may be extended to find the integral of a differential consisting of any number of terms, like $Mdx + Ndy + Pdz +$, &c., by adding to the integral $\int Mdx$ taken relatively to x, the integral of all the terms in Ndy which are independent of x, and then adding the integral of all the terms of Pdz that are independent of either x or y (or both of them), and so on to any required extent. Thus, the integral of

$$\frac{ydx}{z} + \frac{(x + 2ay)\,dy}{z} - \frac{(xy + ay^2)}{z^2}\,dz$$

gives
$$\int\frac{ydx}{z} = \frac{y}{z}\int dx = \frac{yx}{z},$$

$$\frac{a}{z}\int 2ydy = \frac{ay^2}{z}, \quad \text{and} \quad (xy + ay^2)\int -\frac{dz}{z^2} = \frac{xy + ay^2}{z},$$

whose sum, corrected by the addition of an arbitrary constant, is $\frac{xy + ay^2}{z} + C$, which expresses the integral as required.

If $Mdx + Ndy + Pdz +$, &c., is the differential of some function of x, y, z, &c., of u, we shall have

$$M = \frac{du}{dx}, \quad N = \frac{du}{dy}, \quad P + \frac{du}{dz}, \text{ \&c.,}$$

which give

$$\frac{dM}{dy} = \frac{d^2u}{dxdy}, \quad \frac{dN}{dx} = \frac{d^2u}{dydx}, \quad \frac{dM}{dz} = \frac{d^2u}{dxdz},$$

and $$\frac{dP}{dx} = \frac{d^2u}{dzdx}, \quad \frac{dN}{dz} = \frac{d^2u}{dydz}, \quad \frac{dP}{dy} = \frac{d^3u}{dzdy}, \text{ \&c.}$$

Because $\frac{d^2u}{dxdy} = \frac{d^2u}{dydx}$, $\frac{d^2u}{dxdz} = \frac{d^2u}{dzdx}$,

and so on (see p. 22), we shall have

$$\frac{dM}{dy} = \frac{dN}{dx}, \quad \frac{dM}{dz} = \frac{dP}{dx}, \quad \frac{dN}{dz} = \frac{dP}{dy}, \text{ \&c.,}$$

for the *Criteria of Integrability* of a differential of the preceding form; which, being supposed to contain n different variables, will give $\frac{n(n-1)}{1.2}$ equations, like the preceding, in the criteria of integrability, since by the known principles of combinations, $\frac{n(n-1)}{1.2}$ shows how often two may be taken out of n different things.

It is hence clear that any differential which satisfies all the $\frac{n(n-1)}{1.2}$ criteria, can be integrated by the preceding method, and its integral will be exact; but if the criteria are not all satisfied, the integral can not be found, and must be incomplete or inexact; hence the importance of examining the conditions of integrability before we proceed to integrate the equation, becomes too evident to require any further notice.

(5.) Supposing $Adx + Bdy + Cdz +$, &c., to be an exact differential, or one that satisfies all the criteria of integrability, and, at the same time, suppose each of its coefficients, A, B, C, &c., to be of n dimensions in terms of its variables, x, y, z, &c., or, which is the same, suppose the equation to be homogeneous, the degree of homogeneity being n; then, we propose to show that its integral $= \frac{Ax + By + Cz + \&c.}{n+1}$, provided n is different from -1.

Thus, since y, z, &c., may clearly be expressed by xy', xz',

&c., because the differential may evidently be supposed to have been obtained by regarding x alone as variable, it must be expressed by the form $\mathrm{A}dx + \mathrm{B}y'dx + \mathrm{C}z'dx +$, &c.

Because, from the nature of homogeneity, each term of this differential must be supposed to contain the factor $x^n dx$, which, integrated by the rule at p. 254, gives $\dfrac{x^{n+1}}{n+1}$ for the common variable factor of the terms of the integral; consequently, the integral must evidently be expressed by

$$\frac{\mathrm{A}x + \mathrm{B}xy' + \mathrm{C}xz' + \&c.}{n+1} + \text{const.},$$

or its equivalent, $\dfrac{\mathrm{A}x + \mathrm{B}y + \mathrm{C}z + \&c.}{n+1} + \mathrm{C}$,

C being the constant.

It may be noticed that if $n = -1$, the integral

$$\int x^n\,dx = \int \frac{dx}{x} = \log x;$$

consequently, when $n = -1$, it results that $\log x$ must be a factor of the integral of

$$\mathrm{A}dx + \mathrm{B}y'dx + \mathrm{C}z'dz +, \&c.$$

Hence, when n, called the *index of homogeneity*, is different from -1, change dx, dy, dz, &c., severally into x, y, z, &c., in the differential $\mathrm{A}dx + \mathrm{B}dy + \mathrm{C}dz +$, &c., divide the result by the index of homogeneity, increased by unity, and add an arbitrary constant to the quotient for the integral.

EXAMPLES.

1. To find the integral of $(3x^2 + 2axy)\,dx + (ax^2 + 3y^2)\,dy$.

Here the index of homogeneity is clearly 2, being the sum of the indices of x and y in each term of the differential;

consequently, since the differential is clearly integrable, by changing the differentials dx and dy into x and y, we have $(3x^2 + 2axy)\,x + (ax^2 + 3y^2)\,y$. Performing the requisite multiplications, and uniting like terms of the products, we have $3x^3 + 3ax^2y + 3y^3$, which, divided by $2 + 1 = 3$, gives

$$\frac{3x^3 + 3ax^2y + 3y^3}{3} = x^3 + ax^2y + y^3,$$

and adding the constant C to this, we have $x^3 + ax^2y + y^3 + C$ for the integral of the proposed differential.

2. To integrate $(3x^2 + 2bxy - 3y^2)\,dx + (bx^2 - 6xy + 3cy^2)\,dy$.

This being both integrable and homogeneous, we have, as before,

$$\frac{3x^3 + 2bx^2y - 3xy^2 + bx^2y - 6xy^2 + 3cy^3}{3} + C =$$

$$x^3 + bx^2y - 3xy^2 + cy^3 + C$$

for the integral.

3. To integrate $(2y^2x + 3y^3)\,dx + (2x^2y + 9xy^2 + 8y^3)\,dy$.

The answer is $\qquad y^2x^2 + 3y^3x + 2y^4 + C$.

4. To integrate $\dfrac{y\,dx}{z} + \dfrac{(x - 2y)\,dy}{z} + \dfrac{(y^2 - xy)\,dz}{z^2}$.

Since the indices of x and y are positive, while those of z, in the denominators, are to be considered as negative, it is manifest that the index of homogeneity is naught. Hence, it is easy to perceive that the integral is expressed by

$$\frac{xy - y^2}{z} + C.$$

5. To integrate the integrable and homogeneous differential

$$\frac{dx}{\sqrt{(x^2 + y^2)}} + \left(1 - \frac{x}{\sqrt{(x^2 + y^2)}}\right)\frac{dy}{y}.$$

By putting $y = xy'$, the differential is readily reduced to

$$\frac{dx}{x\sqrt{(1+y'^2)}} + \frac{dx}{x} - \frac{dx}{x\sqrt{(1+y'^2)}} = \frac{dx}{x},$$

whose integral may be expressed by

$$\log x + \log C = \log Cx.$$

If $C = C'[1 + \sqrt{(1+y'^2)}]$, we have

$$\log Cx = \log C'[x + x\sqrt{(1+y'^2)}] = \log C'[x + \sqrt{(x^2+y^2)}],$$

which is the well-known form of the integral as determined by the ordinary process of integration; noticing, that the integral appears under quite an undetermined form, on account of the terms that have destroyed each other, agreeably to what is said at pp. 445 and 446, the index of homogeneity in this example being -1.

6. To integrate the integrable and homogeneous differential

$$\frac{xdy}{x^2+y^2} - \frac{ydx}{x^2+y^2}.$$

Here, the index of homogeneity is -1, and the differential is readily reduced to

$$\frac{y'dx}{x(1+y'^2)} - \frac{y'dx}{x(1+y'^2)},$$

whose terms destroy each other, and have the differential $\frac{dx}{x}$ for a common factor; consequently, it is clear that the integral is here under a more undetermined form than in the preceding example. It is hence clear that such integrals as these ought to be avoided as much as possible.

(6.) We will now show how, according to the preceding principles, to integrate a differential expression of the form

$$Qdx^2 + Rdxdy + Sdy^2,$$

in which dx and dy are supposed to be constant or invariable, and x and y are regarded as independent variables; then, because each term of the expression contains two dimensions of the differentials, it is said to be of the second order of differentials, or of the same order as that of the differential of a differential, and in dimensions the expression is said to be of the second degree, or of two dimensions. (See Lacroix's "Calcul Intégral," p. ●32.)

It is easy to perceive that we may consider Qdx^2 and $S\,dy^2$ to have been derived from $\int Qdx$ and $\int Sdy$ by taking their differentials, regarding x and y as separately variable in the expressions; consequently, the proposed differential may be supposed to have been obtained from taking the differentials of the differential $dx\int Qdx + dy\int Sdy$, on the supposition of the constancy of dx and dy, while the first integral is taken on the supposition of the constancy of y, and the second supposing x to be constant.

Hence, by taking the differential of this assumed expression by considering x and y both to vary, and by differentiating under the sign $\int$, according to the rule of Leibnitz, given on page 440, we shall have

$$Qdx^2 + dxdy\int\frac{dQ}{dy}\,dx + dydx\int\frac{dS}{dx}\,dy + Sdy^2;$$

which must clearly be identical with the proposed differential, and thence $\int\frac{dQ}{dy}\,dx + \int\frac{dS}{dx}\,dy = R.$

Differentiating the members of this equation with regard to x alone as variable, and differentiating the second term under the sign $\int$, by the rule of Leibnitz, we shall have

$$\frac{dQ}{dy}\,dx + dx\int\frac{d^2S}{dx^2}\,dy = \frac{dR}{dx}\,dx,$$

or we have $$\frac{dQ}{dy} + \int \frac{d^2S}{dx^2}\, dy = \frac{dR}{dx};$$

and removing the sign $\int$, by differentiating the members of this, regarding y alone as variable, we have

$$\frac{d^2Q}{dy^2} + \frac{d^2S}{dx^2} = \frac{d^2R}{dxdy},$$

which is the same as $\frac{d^2R}{dydx}$ for the condition of integrability of the proposed differential.

(7.) We now propose to show how to find the integral of a differential expression of the form $Pd^2y + Qdx^2$, given by Lacroix at p. 234 of his work, in which x is the independent variable, and y is regarded as being a function of x, and P and Q are supposed to be functions of x, y, dx, dy.

Putting $dy = pdx$, and taking their differentials, regarding dx as being invariable, which we clearly may do, we have $d^2y = dpdx$; which, substituted for d^2y, reduces the given differential to the form $(Pdp + Qdx)\,dx$, which may evidently, as in Lacroix, be represented by the more general form $(Mdp + Ndx)\,dx^n$, whose integral ought evidently to be of the form udx^n; or we must have $u = \int Mdp + V$, supposing the integral to be taken with reference to p, regarding x and y as being constants, and V as being a function of them. Differentiating the members of this equation, regarding M as being a function of x and y, observing the rule of Leibnitz for differentiating under the sign $\int$, we shall have

$$du = Mdp + dx\int \frac{dM}{dx}\, dp + dy\int \frac{dM}{dy}\, dp + \frac{dV}{dx}\, dx + \frac{dV}{dy}\, dy,$$

which, compared to $Mdp + Ndx$, gives

$$N = \int \frac{dM}{dx}\, dp + p \int \frac{dM}{dy}\, dp + \frac{dV}{dx} + \frac{dV}{dy}\, p,$$

which must be an identical equation.

To remove the sign $\int$, we differentiate this twice successively with reference to p, and thence, since V does not contain p, get

$$\frac{dN}{dp} = \frac{dM}{dx} + \int \frac{dM}{dy}\, dp + \frac{dM}{dy}\, p + \frac{dV}{dy},$$

and $$\frac{d^2 N}{dp^2} = \frac{d^2 M}{dx dp} + 2\, \frac{dM}{dy} + \frac{d^2 M}{dy dp}\, p \quad \ldots\ldots\ldots \quad (1),$$

an equation freed from V, that must be satisfied. Hence,

$$\frac{dV}{dy} = \frac{dN}{dp} - \frac{dM}{dx} - \frac{dM}{dy}\, p - \int \frac{dM}{dy}\, dp,$$

and $$\frac{dV}{dx} = N - \frac{dN}{dp}\, p + \frac{dM}{dx}\, p + \frac{dM}{dy}\, p^2 - \int \frac{dM}{dx}\, dp;$$

and since the differential of the first of these with reference to x equals that of the second with reference to y, we have

$$\frac{dN}{dy} - \frac{d^2 N}{dp dx} - \frac{d^2 N}{dp dy} p + \frac{d^2 M}{dx^2} + 2\, \frac{d^2 M}{dx dy} p + \frac{d^2 M}{dy^2}\, p^2 = 0 \ldots (2).$$

When a proposed differential satisfies (1) and (2), by substituting the values of $\frac{dV}{dx}$ and $\frac{dV}{dy}$ in

$$du = M dp + dx \int \frac{dM}{dx}\, dp + dy \int \frac{dM}{dy}\, dp + \frac{dV}{dx}\, dx + \frac{dV}{dy}\, dy,$$

we shall get

$$du = M dp + \left(N - \frac{dN}{dp}\, p + \frac{dM}{dx}\, p + \frac{dM}{dy}\, p^2\right) dx + \left(\frac{dN}{dp} - \frac{dM}{dx} - \frac{dM}{dy}\, p\right) dy,$$

which is freed from $\int$ and under the form of a differential of p, x, and y, whose integral can clearly be found by the rule in (4), at p. 444.

Thus, to find the integral of

$$(2xydy + x^2 ydx)\, d^2y + xdy^3 + (y + x^2)\, dy^2dx + (2 + 3y)\, xydydx^2 + y^3dx^3$$

from what has been done, we put pdx and $dpdx$ for dy and d^2y, and thence get

$$M = 2xyp + x^2y,\ N = xp^3 + (y + x^2)\, p^2 + (2 + 3y)\, xyp + y^3,$$

which give

$$\frac{d^2N}{dp^2} = 6xp + 2\,(y + x^2)\ \frac{d^2M}{dxdp} = 2y\ \frac{2dM}{dy}$$
$$= 4xp + 2x^2,\ \frac{d^2M}{dydp}\,p = 2xp,$$

which will satisfy (2); consequently, the expression is an exact differential, which is reducible to the form

$$du = (2xyp + x^2y)\, dp + (yp^2 + 2xyp + y^3)\, dx + (xp^2 + x^2p + 3xy^2)\, dy.$$

The integrals of the first term of this relative to p, and those of the two last terms relative to x and y, by omitting the terms containing p in them, when added, give

$$xyp^2 + x^2yp + xy^3 + C$$

for the value of u; consequently, since the sought integral evidently has the integral $= udx^2$, we shall have

$$udx^2 = xydy^2 + x^2ydydx + xy^3dx^2 + Cdx^2$$

for the required integral. It is easy to see that we may, in much the same way, proceed to determine the integral of any differential expression between x and y, when it is of any order of differentials greater than the first.

CONCLUDING REMARKS.—Because the differential

$$\frac{A(r-x)\,dx}{(r^2-2rx+b^2)^{\frac{3}{2}}} = -Ad\,\frac{\dfrac{dx}{(r^2-2rx+b^2)^{\frac{1}{2}}}}{dr}$$

$$= -Ad\,\frac{\dfrac{(r^2-2rx+b^2)^{-\frac{1}{2}}rdx}{r}}{dr},$$

we thence get

$$A\int\frac{(r-x)\,dx}{(r^2-2nx+b^2)^{\frac{3}{2}}} = -Ad\,\frac{\int\dfrac{(r^2-2rx+b^2)^{-\frac{1}{2}}rdx}{r}}{dr}$$

$$= Ad\,\frac{\dfrac{(r^2-2rx+b^2)^{\frac{1}{2}}}{r}}{dr} + C$$

for the integral.

It is hence easy to perceive how the forms of differentials may be sometimes changed, so as greatly to facilitate their integration, by taking the differential of them with reference to a constant in them.

SECTION VI.

INTEGRATION OF DIFFERENTIAL EQUATIONS OF THE FIRST ORDER AND DEGREE, BETWEEN TWO VARIABLES.

(1.) It is manifest that a differential equation between any number of variables, when the variables are separated from each other, is such that the integral can always be found; and if the terms of an equation are of an integrable form, it may evidently be integrated by the methods given in Section V.

(2.) If we have an equation of the form $X dy + Y dx = 0$, between x and y, such that X is a function of x alone, and Y a function of y alone, then, dividing the equation by XY the product of the differential coefficients, it is reduced to $\frac{dy}{Y} + \frac{dx}{X} = 0$, which is clearly an integrable form, or such that the integral $\int\frac{dy}{Y} + \int\frac{dx}{X} = 0$ can be found.

Thus, the particular differential equation

$$(x + 1)^2 dy = (y + 1)^3 dx$$

is reducible to $$\frac{dy}{(y + 1)^3} = \frac{dx}{(x + 1)^2},$$

whose integral is $$\frac{1}{2(y + 1)^2} = \frac{1}{x + 1} + C.$$

(3.) Similarly, the differential form

$$XY dy + X_1 Y_1 dx = 0,$$

divided by the partial product XY, of the differential coefficients, becomes $\frac{Ydy}{Y_1} + \frac{X_1 dx}{X} = 0,$

in which the variables are separated, and it is clearly an integrable form, or such that the integral

$$\int \frac{Ydy}{Y} + \int \frac{Xdx}{X} = 0$$

can be found.

Thus, the particular differential equation

$$(x + 1) y^2 dx = (y^2 + 1) xdy = 0,$$

divided by xy^2, becomes

$$\left(1 + \frac{1}{x}\right) dx = \left(1 + \frac{1}{y^2}\right) dy;$$

which is clearly an integrable form, the integral being

$$x + \log x = y - \frac{1}{y} + C.$$

(4.) The equation $dy + Pydx = Qdx,$

sometimes called a *linear equation*, can have its variables separated by assuming

$$dy + Pydx = 0, \quad \text{which gives} \quad \frac{dy}{y} = - Pdx,$$

whose integral may evidently be expressed by

$$\log y - \log C = \log \frac{y}{C} = - \int Pdx,$$

or using e for the hyperbolic base, $y = Ce^{-\int Pdx}$. To adapt this to the proposed question, we may suppose C to vary; consequently, by taking the differential of y on this supposition, we shall have

$$dy = - Ce^{-\int Pdx} dx + dCe^{-\int Pdx}.$$

By substituting y and dy in the proposed equation, and erasing the terms that destroy each other, we have

$$dCe^{-\int Pdx} = Qdx, \quad \text{or} \quad dC = e^{\int Pdx} Q\,dx,$$

whose integral gives $C = \int e^{\int Pdx} Qdx + C'$. Hence, from the substitution of this value of C in that of y, it becomes

$$y = e^{-\int Pdx}\left(\int e^{\int Pdx} Qdx + C'\right)$$

for the integral of the proposed equation.

REMARKS.—Hence, the integral obtained from a very simple case of the proposed differential equation, by the variation of the arbitrary constant, has enabled us to find the integral when taken in its utmost extension.

Otherwise.—By assuming $y = Xz$, we shall get

$$dy = zdX + Xdz,$$

which values of y and dy, substituted in the proposed equation, reduce it to

$$zdX + Xdz + PXzdx = Qdx,$$

in which X being arbitrary, we may assume

$$dz + Pzdx = 0 \quad \text{or} \quad z = e^{-\int Pdx},$$

and thence get

$$dX = \frac{Qdx}{z} = z^{-1}Qdx = e^{\int Pdx} Qdx,$$

whose integral is $X = \int e^{\int Pdx} Qdx + C'$.

Hence, from the substitution of these values of X and z, we shall have

$$y = Xz = e^{-\int Pdx}\left(\int e^{\int Pdx} Qdx + C'\right)$$

for the integral, the same as found by the preceding method.

REMARK.—This method of integration has been taken from p. 254 of Lacroix's "Calcul Intégral."

(5.) The more general differential equation

$$dy + Pydx = Qy^{n+1}dx$$

can readily be reduced to the preceding form.

For by multiplying its terms by $-\frac{n}{y^{n+1}}$, it becomes

$$-\frac{ndy}{y^{n+1}} - \frac{nPdx}{y^n} = -nQdx;$$

which, by putting $z = \frac{1}{y^n}$, becomes

$$dz - nPzdx = -nQdx,$$

which is of like form to the differential equation in (4). Hence, by putting $-nP$ and $-nQ$ for P and Q in the integral in (4), we shall have

$$z = e^{n\int Pdx}\left(-n\int e^{-n\int Pdx}Qdx + C'\right)$$

for the integral of the preceding equation, and thence we get y. (See p. 192 of Young's "Integral Calculus.")

To illustrate the preceding formulas, take the following

EXAMPLES.

1. To find the integral of $dy + ydx = ax^2dx$.

Comparing the equation to that in (4), we have

$$P = 1 \quad \text{and} \quad Q = ax^2, \quad \text{and thence} \quad \int Pdx = x,$$

which reduces $e^{\int Pdx}$ to e^x, and

$$\int e^{\int Pdx}Qdx + C' \quad \text{reduces to} \quad a\int e^x x^2 dx + C;$$

whose integral, being found by integrating by parts, gives

$$a\int e^x x^2\,dx + C' = ae^x(x^2 - 2x + 2) + C'.$$

Hence, from $y = e^{-\int Pdx}\left(\int e^{\int Pdx}\,Q dx + C'\right)$,

we get $$y = a(x^2 - 2x + 2) + C'e^{-x}$$

for the sought integral.

2. To find the integral of $dy + ydx = ax^n\,dx$.

Here we have $P = 1$, $Q = ax^n$, and thence

$$e^{\int Pdx} = e^x \int e^{\int Pdx}\,Qdx + C' = a\int e^x x^n\,dx + C',$$

which, integrated by parts, as before, becomes

$$e^x a\,[x^n - nx^{n-1} + n(n-1)\,x^{n-2} - \&c.] + C';$$

consequently, we shall have

$$\begin{aligned} y &= e^{-\int Pdx}\,(e^{\int Pdx}\,Qdx + C') \\ &= Q\,[x^n - nx^{n-1} + n(n-1)\,x^{n-2} - \&c.] + C'e^{-x} \end{aligned}$$

for the required integral.

3. To find the integral of $dy + y\dfrac{xdx}{1+x^2} = \dfrac{ax}{1+x^2}\,dx.$

Here we have

$$P = \frac{x}{1+x^2}, \quad Q = \frac{ax}{1+x^2},$$

$$\int Pdx = \log(1+x^2)^{\frac{1}{2}}, \quad e^{\int Pdx} = e^{\log\sqrt{(1+x^2)}},$$

and $$\int e^{\int Pdx}\,Qdx + C' = Q\int e^{\log\sqrt{(1+x^2)}}\,\frac{xdx}{1+x^2} + C'$$
$$= Qe^{\log\sqrt{(1+x^2)}} + C';$$

consequently,

$$y = e^{-\int Pdx}\,x\left(\int e^{\int Pdx}\,Qdx + C'\right) = Q + C'e^{-\log\sqrt{(1+x^2)}}$$

is the required integral.

4. To find the integral of $dy + ydx = y^3 xdx$.

Here, from the formula in (5), we shall have

$$P = 1, \quad Q = x, \quad n = 2, \quad \int P dx = x,$$

and $$\int -ne^{-n\int Pdx} Qdx + C' = \int -2e^{-2x} xdx + C'$$
$$= e^{-2x}\left(x + \frac{1}{2}\right) + C';$$

consequently, we shall have

$$z = \frac{1}{y^2} = \left(\frac{1}{2} + x\right) + C'e^{2x}$$

for the required integral.

5. To find the integral of $dy + y\dfrac{xdx}{1-x^2} = y^2\dfrac{xdx}{1-x^2}$.

Here $P = \dfrac{x}{1-x^2}$, $\quad Q = \dfrac{x}{1-x^2}$, $\quad$ and $\quad n = 1$;
thence we have

$$\int Pdx = -\log\sqrt{(1-x^2)} \quad \text{and} \quad e^{n\int Pdx} = e^{-\log\sqrt{(1-x^2)}}.$$

Hence, we shall have

$$-n\int e^{-n\int Pdx} Qdx + C' = -\int e^{\log\sqrt{(1-x^2)}}\frac{xdx}{1-x^2} + C'$$
$$= e^{\log\sqrt{(1-x^2)}} + C',$$

which gives $\quad z = \dfrac{1}{y} = 1 + C'\, e^{\log\sqrt{(1-x^2)}}$

for the right integral.

6. To find the integral of $dy - \dfrac{yxdx}{1+x^2} = \dfrac{adx}{1+x^2}$.

Here $$P = -\frac{x}{1+x^2}, \quad Q = \frac{a}{1+x^2},$$
$$\int Pdx = -\log\sqrt{(1+x^2)},$$

and $$e^{\int Pdx} = e^{-\log\sqrt{(1+x^2)}} = \frac{1}{\sqrt{(1+x^2)}},$$ and thence

$$\int e^{\int \mathrm{P}dx}\mathrm{Q}dx + \mathrm{C}' = a\int \frac{dx}{(1+x^2)^{\frac{3}{2}}} + \mathrm{C}' = \frac{ax}{\sqrt{1+x^2}} + \mathrm{C}';$$

consequently, $y = ax + \mathrm{C}'\sqrt{1+x^2}$ is the integral.

7. To find the integral of $dy + \dfrac{yxdx}{1-x^2} = y^{\frac{1}{2}}xdx.$

Here $\mathrm{P} = \dfrac{x}{1-x^2}$, $\mathrm{Q} = x$, $n = -\dfrac{1}{2}$,

$$\int \mathrm{P}dx = -\log(1-x^2)^{\frac{1}{2}}, \quad n\int \mathrm{P}dx = \log(1-x^2)^{\frac{1}{4}},$$

and $$e^{n\int \mathrm{P}dx} = e^{\log(1-x^2)^{\frac{1}{4}}} = (1-x^2)^{\frac{1}{4}},$$

from the nature of numbers and their hyperbolic logarithms. We also have

$$-n\int e^{-n\int \mathrm{P}dx}\mathrm{Q}dx + \mathrm{C}' = \frac{1}{2}\int \frac{xdx}{(1-x^2)^{\frac{1}{4}}} = -\frac{1}{3}(1-x^2)^{\frac{3}{4}} + \mathrm{C}';$$

consequently, from $z = \dfrac{1}{y^n} = \dfrac{1}{y^{-\frac{1}{2}}} = y^{\frac{1}{2}}$,

since $$z = e^{n\int \mathrm{P}dx}\left(-n\int e^{-n\int \mathrm{P}dx}\mathrm{Q}dx + \mathrm{C}'\right)$$

$$= (1-x^2)^{\frac{1}{4}}\left(-\frac{1}{3}(1-x^2)^{\frac{3}{4}} + \mathrm{C}'\right),$$

we shall have $y^{\frac{1}{2}} = \mathrm{C}'(1-x^2)^{\frac{1}{4}} - \dfrac{1}{3}(1-x^2)$

for the required integral.

(6.) If $\mathrm{M}dx + \mathrm{N}dy = 0$ is a homogeneous function of x and y of the degree n, its variables x and y may be separated.

For if we divide M and N by x^n, it is manifest that the equation will be reduced to the form

$$f\left(\frac{y}{x}\right)dx + f'\left(\frac{y}{x}\right)dy = 0,$$

since it is clear that the dimensions of y in the numerators of the quotients equal those of x in the corresponding denominators. If we put $\frac{y}{x} = z$ or $y = xz$, we have

$$dy = zdx + xdz,$$

and thence our equation is reduced to

$$f(z)\,dx + f'(z)\,(zdx + xdz) = 0,$$

or
$$[f(z) + zf'(z)]\,dx = -xf'(z)\,dz,$$

or its equivalent $\frac{dx}{x} = -\frac{f'(z)\,dz}{f(z) + zf'(z)}$,

in which the variables are separated; consequently, we shall have
$$\log x = -\int \frac{f'(z)\,dz}{f(z) + zf'(z)}.$$

EXAMPLES.

1. To find the integral of $(x^2 + yx)\,dy = (xy + y^2)\,dx$.

Dividing by x^2, we have

$$\left(1 + \frac{y}{x}\right) dy = \left(\frac{y}{x} + \frac{y^2}{x^2}\right) dx, \quad \text{or} \quad (1 + z)\,dy = (z + z^2)\,dx,$$

or
$$dy = \frac{ydx}{x}, \quad \text{or} \quad \frac{dy}{y} = \frac{dx}{x},$$

which gives $\log y = \log cx$, or $y = cx$; which results immediately from the proposed equation, by erasing the factor $x + y$ that is common to its members.

2. To find the integral of $[y + \sqrt{(x^2 - y^2)}]\,dx = xdy$.

Dividing by x, we have

$$z + \sqrt{1 - z^2} = \frac{dy}{dx}, \quad f(z) = z + \sqrt{1 - z^2}, \quad \text{and } f'(z) = -1,$$

and thence, from the formula, we have

$$\log x = \int \frac{dz}{\sqrt{1-z^2}} = \sin^{-1} z + C = \sin^{-1}\frac{y}{x} + C.$$

3. To find the integral of $(yx + y^2)\,dx = (x^2 - xy)\,dy$.

Dividing by x^2, we have

$$(z + z^2)\,dx = (1 - z)\,dy;$$

and thence, since $z + z^2 = f(z)$ and $1 - z = -f'(z)$, we shall have, by the formula,

$$2 \log x = \int dz \left(\frac{1}{z^2} - \frac{1}{z}\right), \quad \text{or} \quad \log x^2 + \log z + \frac{1}{z} = C,$$

or

$$\frac{x}{y} + \log xy = C,$$

as required.

4. To integrate $(\sqrt{x^2 + y^2} + y)\,dx = x\,dy$.

Dividing by x, we have

$$(\sqrt{1 + z^2} + z)\,dx = dy.$$

Hence, by the formula, we shall have

$$\log x = \int \frac{dz}{\sqrt{1 + z^2}} = \log c(z + \sqrt{z^2 + 1}),$$

or

$$x^2 = c\,[y + \sqrt{(y^2 + x^2)}],$$

which may evidently be changed to the form

$$(x^2 - cy)^2 = c^2\,(y^2 + x^2),$$

or its equivalent

$$x^2 = 2cy + c^2.$$

5. To integrate $(x + zy)\,dx + y\,dy = 0$.

Here

$$(1 + 2z)\,dx + z\,dy = 0,$$

$$f(z) = 1 + 2z, \quad \text{and} \quad f'(z) = z;$$

and thence by the formula we shall have

$$\log x = -\int \frac{z\,dz}{(z+1)^2} = -\log(1+z) - \frac{1}{1+z} + \text{C},$$

or we have $$\log(x+y) + \frac{x}{x+y} = \text{C}.$$

(7.) Equations between x and y may sometimes be made homogeneous by certain substitutions, and thence their integrals may be found. Thus, if in

$$(mx + ny + p)\,dx + (ax + by + c)\,dy = 0,$$

we put $$x = x' + \text{A}, \quad y = y' + \text{B},$$

and assume

$$\text{A}m + \text{B}n + p = 0, \quad \text{A}a + \text{B}b + c = 0,$$

we shall have the homogeneous differential equation

$$(mx' + ny')\,dx' + (ax' + by')\,dy' = 0,$$

whose integral can thence be found.

Solving the equations

$$\text{A}m + \text{B}n + p = 0, \quad \text{A}a + \text{B}b + c = 0,$$

we have $\text{A} = \dfrac{cn - bp}{mb - an}$ and $\text{B} = \dfrac{ap - mc}{mb - an}$,

which give the values of A and B when $mb - an$ is different from naught; but when $mb - an = 0$, we have $b = \dfrac{an}{m}$, which reduces the proposed equation to

$$(mx + ny + p)\,dx + (ax + by + c)\,dy =$$

$$(mx + ny + p)\,dx + \frac{a}{m}\left(nx + ny + \frac{mc}{a}\right)dy = 0,$$

or $$p\,dx + c\,dy + (mx + ny).\left(dx + \frac{a}{m}\,dy\right) = 0;$$

which, by putting $mx + ny = z$, gives

$$m dx + n dy = dz \quad \text{and} \quad p dx + c dy + z\left(dx + \frac{a}{m} dy\right) = 0,$$

or we have $$(p + z)\, dx + \frac{mc + az}{m} dy = 0.$$

Hence, since $dy = \dfrac{dz - m dx}{n}$, we readily get

$$dx + \frac{(mc + az)\, dz}{mnp - m^2 c + (mn - am) z} = 0,$$

in which the variables are separated; and if $n = a$, this reduces to the very simple form

$$dx + \frac{(mc + az)\, dz}{mnp - m^2 c} = 0.$$

(See Lacroix, p. 253.)

(8.) Particular cases of integrability of differential equations between x and y may often be discovered by reducing them to homogeneity.

To illustrate this, let there be taken the equation

$$dy + by^2 dx = ax^m dx,$$

called the *equation of Riccati.*

1. If $m = 0$, the equation is equivalent to $dx = \dfrac{dy}{a - by^2}$, in which the variables are separated, and of course it is integrable. Indeed, since

$$a - by^2 = (a^{\frac{1}{2}} + b^{\frac{1}{2}} y)(a^{\frac{1}{2}} - b^{\frac{1}{2}} y),$$

we easily get

$$2a^{\frac{1}{2}} dx = \frac{dy}{a^{\frac{1}{2}} + b^{\frac{1}{2}} y} + \frac{dy}{a^{\frac{1}{2}} - b^{\frac{1}{2}} y},$$

whose integral is

$$2a^{\frac{1}{2}} x = \frac{1}{b^{\frac{1}{2}}} \log (a^{\frac{1}{2}} + b^{\frac{1}{2}} y) - \frac{1}{b^{\frac{1}{2}}} \log (a^{\frac{1}{2}} - b^{\frac{1}{2}} y) + C.$$

2. If m is different from naught, we may put $y = z^k$, and thence get $dy = kz^{k-1}dz$; consequently, from the substitution of the values of y and dy in the proposed equation, we have

$$kzdz^{k-1} + bz^{2k}dx = ax^m dx.$$

To make this a homogeneous equation, we must equate the exponents of z and x, and we shall have $k - 1 = 2k = m$, or $k = -1$ and $m = -2$; consequently, the equation

$$dy + by^2dx = ax^m dx$$

becomes integrable when we put z^{-1} for y, and $-z$ for m, and is reduced to

$$-z^{-2}dz + bz^{-2}dx = ax^{-2}dx,$$

or its equivalent $\quad -\dfrac{dz}{z^2} + \dfrac{bdx}{z^2} = \dfrac{adx}{x^2}.$

3. If, with Lacroix, at p. 256 of his "Calcul Intégral," we put $y = \dfrac{y'}{x^2} + \dfrac{1}{bx}$, we shall have

$$dy = \frac{dy'}{x^2} - \frac{2xy'dx}{x^4} - \frac{dx}{bx^2} \quad \text{and} \quad y^2 = \frac{y'^2}{x^4} + \frac{2y'}{bx^3} + \frac{1}{b^2x^2},$$

and thence we shall have

$$dy + by^2dx = \frac{dy'}{x^2} + \frac{by'^2dx}{x^4};$$

or, since $\quad dy + by^2dx = ax^m dx,$

we shall have $\quad \dfrac{dy'}{x^2} + \dfrac{by'^2dx}{x^4} = ax^m dx,$

or $\quad dy' + by'^2\dfrac{dx}{x^2} = ax^{m+2}dx;$

which, by putting $x = \dfrac{1}{x'}$, becomes

$$dy' - by'^2dx' = -ax'^{-m-4}dx',$$

or putting $-y'$ for y', we shall have

$$dy' + by'^2dx' = ax'^{-m-4},$$

which is an equation of the form

$$dy + by^2dx = ax^m dx;$$

and becomes integrable, as before, when $m + 4 = 0$, or when $m = -4$, and is obtained immediately from

$$dy + by^2dx = ax^m dx,$$

by putting $y = -\frac{y'}{x^2} + \frac{1}{bx}$ or $y = -y'x'^2 + \frac{x'}{b}$, when x' is put for $\frac{1}{x}$.

It is hence clear that the equations

$dy + by^2dx = ax^{-m-4}dx$ and $dy' + by'^2dx' = ax'^{-m-4}dx'$,

are of such a nature, that if in the first we put

$$x = \frac{1}{x'}, \text{ and } y = -y'x'^2 + \frac{x'}{b}:$$

it will be changed into the second; and that if in the second we put

$$x' = \frac{1}{x} \text{ and } y' = -yx^2 + \frac{x}{b}:$$

it will be changed into the first; consequently, either of the equations is a transformation of the other.

4. Resuming the equation

$$dy + by^2 dx = ax^m dx,$$

and putting $y = \pm \frac{1}{y'}$, we have $dy = \mp \frac{dy'}{y'^2}$; and thence we get

$$\mp \frac{dy'}{y'^2} + \frac{bdx}{y'^2} = ax^m dx, \quad \text{or} \quad \mp dy' + bdx = ay'^2 x^m dx.$$

If we put $x^{m+1} = x'$, we have

$$x = x'^{\frac{1}{m+1}}, \qquad x^m\, dx = \frac{dx'}{m+1};$$

and thence the preceding equation is easily reduced to

$$\mp\, dy' + \frac{b}{m+1}\, x'^{\frac{-m}{m+1}}\, dx' = \frac{a}{m+1}\, y'^2\, dx',$$

or to $$dy' \pm \frac{a}{m+1}\, y'^2\, dx' = \pm \frac{b}{m+1}\, x'^{\frac{-m}{m+1}}\, dx'.$$

It is manifest that if $m = -4$ in the equation of Riccati, that it will be integrable, and thence

$$dy' \pm \frac{a}{m+1}\, y'^2\, dx' = \pm \frac{b}{m+1}\, x'^{\frac{-m}{m+1}}\, dx'$$

derived from it, and having the same form, by putting

$$y = \pm \frac{1}{y'} \quad \text{and} \quad x^{m+1} = x',$$

must also be integrable; that is to say, the equation

$$dy + by^2\, dx = ax^{-4}\, dx$$

being integrable, it follows that

$$dy' \pm \frac{a}{-3}\, y'^2\, dx = \pm \frac{b}{-3}\, x'^{-\frac{4}{3}}\, dx'$$

must also be integrable, and thence, by putting

$$-m-4 = -\frac{4}{3} \quad \text{or} \quad m = \frac{4}{3} - 4 = -\frac{8}{3},$$

is the value of m for another integrable case; and putting $-\frac{8}{3}$ for m in the equation

$$dy' \pm \frac{a}{m+1}\, y'^2\, dx' = \pm \frac{b}{m+1}\, x'^{\frac{-m}{m+1}}\, dx',$$

we have $m = -\frac{8}{5}$ for the value of m in another integrable case of the equation of Riccati, and so on; noticing, that the general form of the exponent m, when 0 and -2 are not included, is $m = -\dfrac{4q}{2q \pm 1}$, which is called the *Criterion of Integrability of Riccati's Equation*, q being any number in the series 1, 2, 3, 4, &c.

It may be noticed, that all the terms that result from taking $-$ for $\pm$ in the denominator of the criterion, must be considered as resulting from the equation

$$dy + by^2\,dx = ax^{-m-4};$$

while those terms that result from taking $+$ for $\pm$ in the denominator of the criterion, must be supposed to have resulted from the equation

$$dy' \pm \frac{a}{m+1}\,y'^2 dx' = \pm\frac{b}{m+1}\,x'^{\frac{-m}{m+1}}.$$

5. To perceive the use of what has been done, take the following

EXAMPLES.

1. To find the integral of $dy + y^2dx = a^2x^{-4}dx$.

Here, by putting $q = 1$ in the criterion, and using $-$ for ± 1 in its denominator, it becomes

$$m = \frac{-4}{2-1} = -4,$$

which agrees with the exponent of x in the right member of the proposed equation, and of course shows the equation to be integrable. To perform the integration we proceed, as at p. 466, by putting

$$x = \frac{1}{x'} \quad \text{and} \quad y = -y'x'^2 + x', \quad \text{since} \quad b = 1,$$

and thence get $dy' + y'^2dx' = a^2x'^{-m}dx,$

as at p. 466; where, by regarding -4, the exponent of x in the right member of the proposed equation, as being equal to $-m-4$ the exponent of x in the equation

$$dy + by^2dx = a^2x^{-m-4}dx,$$

we shall have $m = 0$; and thence the preceding equation reduces to $$dy' + y'^2dx' = a^2dx',$$

which gives

$$dx' = \frac{dy'}{a^2 - y'^2} = \left(\frac{dy'}{a+y'} + \frac{dy'}{a-y'}\right) \div 2a.$$

Integrating this equation, we have

$$2ax' = \log C\,\frac{a+y'}{a-y'}, \quad \text{or} \quad e^{2ax'} \times \frac{a-y'}{a+y'} = C = \text{const.}$$

From $x' = \dfrac{1}{x}$ and $y' = -yx^2 + x$, we get

$$e^{\frac{2a}{x}}\left(\frac{x(xy-1)-a}{x(-xy-1)-a}\right) = C$$

for the required integral.

2. To find the integral of $dy + y^2dx = -a^2x^{-4}dx$.

Putting $x = \dfrac{1}{x}$ and $y = -y'x'^2 + x'$, we get, as in the preceding question, $dy' + y'^2dx' = -a^2dx'$.

Hence $$dx' = -\frac{dy'}{a^2 + y'^2} = -\frac{1}{a}\,\frac{\dfrac{dy'}{a}}{1+\left(\dfrac{y'}{a}\right)^2},$$

whose integral gives $ax' + C = \cot^{-1}\dfrac{y'}{a}$;

or, since $x' = \dfrac{1}{x}$ and $y' = -yx^2 + x$, we have

$$\frac{a}{x} + C = \cot^{-1}\frac{-yx^2 + x}{a}.$$

3. To find the integral of $dy + y^2dx = 2x^{-\frac{4}{3}}dx$.

Here, by putting $q = 1$, and using $+$ for $\pm$ in the denominator of the criterion, we have $m = -\frac{4}{3}$, and of course we must compare the proposed equation to the equation

$$dy' \pm \frac{a}{m+1} y'^2dx' = \pm \frac{b}{m+1} x'^{\frac{-m}{m+1}} dx',$$

given on p. 467; consequently, we shall have

$$\frac{a}{m+1} = 1, \quad \frac{b}{m+1} = 2, \quad \text{and} \quad \frac{m}{m+1} = \frac{4}{3},$$

agreeably to what is said at p. 468; hence,

$$3m = 4m + 4 \quad \text{or} \quad m = -4, \quad a = m + 1 = -3,$$
$$b = 2m + 2 = -6,$$

and thence we get

$$dy'' - 6y''^2, \quad dx'' = -3x''^{-4}dx''.$$

Hence, from the formulas at p. 466, we have

$$dy''' - 6y'''^2dx''' = -3dx''';$$

since $-m - 4$, the exponent of x is here -4, and of course $m = 0$. From this equation we have

$$dx''' = \frac{dy'''}{6\left(y'''^2 - \frac{1}{2}\right)},$$

whose integral is

$$x''' = -\frac{1}{6\sqrt{2}} \log C \frac{y''' + \sqrt{\frac{1}{2}}}{y''' - \sqrt{\frac{1}{2}}}.$$

Because $\quad x''' = \frac{1}{x''}$ and $y''' = -y''x''^2 - \frac{x''}{6}$,

and from the formulas at p. 466,

$$y = \pm \frac{1}{y'}, \qquad x'^{\frac{1}{m+1}} = x'',$$

we here have

$$x''' = \frac{1}{x''}, \quad y'' = \pm \frac{1}{y'}, \quad \text{and} \quad x'' = x'^{\frac{1}{m+1}}.$$

Hence, since $m + 1 = -3$, we have

$$x'' = x'^{-\frac{1}{3}} \quad \text{and} \quad x''' = \frac{1}{x''} = \frac{1}{x'^{-\frac{1}{3}}}:$$

or, since $x = \frac{1}{x'}$, we have $x^{-\frac{1}{3}} = \frac{1}{x'^{-\frac{1}{3}}}$, and thence $x''' = x^{-\frac{1}{3}}$;

and from

$$y''' = -y''x''^2 - \frac{x''}{6}, \quad y'' = \frac{1}{y'}, \quad \text{and} \quad y' = -yx^2 + x,$$

we have

$$y''' = -\frac{x''^2}{-yx^2 + x} - \frac{x''}{6} = -\frac{6 + x^{\frac{2}{3}}(1 - yx)}{6x^{\frac{1}{3}}(1 - yx)}.$$

Hence, from the substitution of the values of x''' and y''' in

$$x''' = -\frac{1}{6\sqrt{2}} \log C \frac{y''' + \sqrt{\frac{1}{2}}}{y''' - \sqrt{\frac{1}{2}}},$$

we shall get

$$x^{-\frac{1}{3}} = -\frac{1}{6\sqrt{2}} \log C \frac{6 - x^{\frac{1}{3}}(3\sqrt{2} - x^{\frac{1}{3}})(1 - yx)}{6 + x^{\frac{1}{3}}(3\sqrt{2} + x^{\frac{1}{3}})(1 - yx)};$$

or $$6\sqrt{2}x^{-\frac{1}{3}} = \log e^{6\sqrt{2}x^{-\frac{1}{3}}}$$

gives $$e^{6\sqrt{2x}^{-\frac{1}{3}}} \left(\frac{6 + x^{\frac{1}{3}}(3\sqrt{2} + x^{\frac{1}{3}})(1 - yx)}{6 - x^{\frac{1}{3}}(3\sqrt{2} - x^{\frac{1}{3}})(1 - yx)}\right) = \text{const.}$$

for the sought integral.

4. To integrate $dy - y^2dx = 2x^{-\frac{4}{3}}dx$.

Comparing the equation to

$$dy' \pm \frac{a}{m+1} y'^2 dx = \frac{b}{m+1} x'^{\frac{-m}{m+1}} dx,$$

we have

$$\frac{a}{m+1} = -1, \quad \frac{b}{m+1} = 2, \quad \text{and} \quad \frac{m}{m+1} = \frac{4}{3},$$

which give $m = -4$, $a = 3$, and $b = -6$, and thence we have

$$dy'' - 6y''^2 dx'' = 3x''^{-4} dx''.$$

Hence, from the formulas at p. 466, we have

$$dy''' - 6y'''^2 dx''' = 3dx''',$$

which gives

$$dx''' = \frac{dy'''}{6\left(y'''^2 + \frac{1}{2}\right)} = \frac{dy''' \sqrt{2}}{3\sqrt{2}\,(1 + 2y'''^2)},$$

whose integral gives

$$3\sqrt{2}x''' = \tan^{-1} y''' \sqrt{2} + C.$$

From

$$x''' = \frac{1}{x''} \quad \text{and} \quad x'' = x'^{\frac{1}{m+1}} = x'^{-\frac{1}{3}} = \frac{1}{x^{-\frac{1}{3}}},$$

we have

$$x''' = x^{\frac{1}{3}};$$

also $y''' = -y''x''^2 \dfrac{x''}{6}$ and $y'' = \dfrac{1}{y'} = \dfrac{1}{-yx^2 + x}$,

we have

$$y''' = -\frac{x^{\frac{2}{3}}}{-yx^2 + x} - \frac{x^{\frac{1}{3}}}{6} = \frac{6}{6x^{\frac{1}{3}}(yx-1)} - \frac{x^{\frac{2}{3}}(yx-1)}{6x^{\frac{1}{3}}(yx-1)}$$

$$= \frac{x^{\frac{2}{3}} - yx^{\frac{5}{3}} + 6}{6x^{\frac{1}{3}}(yx-1)}.$$

Hence, we shall have

$$\frac{3\sqrt{2}}{x^{\frac{1}{3}}} = \tan^{-1} \frac{x^{\frac{2}{3}} - yx^{\frac{5}{3}} + 6}{3\sqrt{2}x^{\frac{1}{3}}(yx-1)} + C$$

for the required integral.

(9.) It may be added, that differential equations may often, by the introduction of new variables and particular processes, be reduced to integrable forms.

1. Thus, to find the integral of

$$\frac{pdx}{x} + \frac{rdy}{y} = \frac{x^m dx}{ay^n},$$

since the integral of the terms

$$\frac{pdx}{x} + \frac{rdy}{y} \quad \text{is} \quad \log x^p y^r,$$

by putting $x^p y^r = z$ we have $y^r = \frac{z}{xp}$ or $y = \left(\frac{z}{x^p}\right)^{\frac{1}{r}}$, and thence $y^n = \left(\frac{z}{x^p}\right)^{\frac{n}{r}}$; consequently, the proposed equation is reduced to $\quad d \log x^p y^r = d \log z = \dfrac{x^{\frac{mr+np}{r}} dx}{z^{\frac{n}{r}}},$

or $\quad \dfrac{z^{\frac{n}{r}} dz}{z} = x^{\frac{mr+np}{r}} dx,\quad$ or $\quad z^{\frac{n}{r}-1} dz = x^{\frac{mr+np}{r}} dx,$

in which the variables are separated. Integrating this, we have

$$\frac{z^{\frac{n}{r}}}{\frac{n}{r}} = \frac{x^{\frac{mr+np+r}{r}}}{\frac{mr+np+r}{r}} + \text{const.},$$

or

$$\frac{az^{\frac{n}{r}}}{n} = \frac{x^{\frac{mr+np+r}{r}}}{mr+np+r} + \text{const.}$$

Restoring the value of z, we have

$$ay^n x^{\frac{np}{r}} = \frac{nx^{\frac{mr+np+2}{r}}}{mr+np+r},$$

which needs no correction, supposing y and x to commence together; dividing the members of this equation by $x^{\frac{np}{r}}$, it is immediately reduced to

$$ay^n = \frac{nx^{m+1}}{mr + np + r}.$$

REMARKS.—The preceding method of finding the integral is analogous to that of Lacroix, at p. 259 of his "Calcul Intégral."

The integral can also be immediately found by multiplying its members by $\frac{n}{r} x^{\frac{np}{r}} y^n$, which gives

$$\frac{np}{r} y^n x^{\frac{np}{r}-1} dx + nx^{\frac{np}{r}} y^{n-1} dy = d\left(x^{\frac{np}{r}} y^n\right) = \frac{n}{ra} x^{\frac{mr+np}{r}} dx;$$

whose integral, as above, is

$$ax^{\frac{np}{r}} y^n = \frac{nx^{\frac{mr+np+r}{r}}}{mr+np+r} \quad \text{or} \quad ay^n = \frac{nx^{\frac{m+1}{r}}}{mr+np+r},$$

supposing the integral to commence with x.

2. To integrate the equation

$$\frac{dy}{y} - \frac{dx}{x} = \frac{x^m dx}{ay\sqrt{n}},$$

we multiply its members by $\frac{y}{x}$, and thence get

$$\frac{dy}{x} - \frac{ydx}{x^2} = \frac{x^{m-1} dx}{a\sqrt{n}},$$

an exact differential. Taking the integral, we have

$$\frac{y}{x} = \frac{x^m}{ma\sqrt{n}} \quad \text{or} \quad y = \frac{x^{m+1}}{ma\sqrt{n}};$$

which needs no correction, supposing the integral to commence with x.

3. To integrate the equation

$$(a+y)\frac{dx}{dy} = x + y - x\frac{dy}{dx};$$

we may clearly, from what is shown at pp. 34 and 35, take the differential of its members by regarding dx as being constant, and shall thence get

$$dx - \frac{d^2ydx}{dy^2}(a+y) = dx + dy - dy - \frac{d^2y}{dx}x;$$

or, by reduction, we shall have

$$\frac{dx}{dy^2}(a+y) = \frac{x}{dx}, \quad \text{or} \quad \frac{dx^2}{x} = \frac{dy^2}{a+y},$$

in which the variables are separated. Hence, we shall have

$$\frac{dy}{(a+y)^{\frac{1}{2}}} = \pm\frac{dx}{x^{\frac{1}{2}}},$$

whose integral gives

$$(a+y)^{\frac{1}{2}} = \pm x^{\frac{1}{2}} + c;$$

or, by squaring,

$$y + a = x \pm 2cx^{\frac{1}{2}} + c^2,$$

which can be further reduced to

$$(y + a - x^{\frac{1}{2}} - c^2)^2 = 4c^2x,$$

which represents the integral of the proposed equation, taken in its most general sense.

4. To find the integral of $ady = ydx - xdx$.

By assuming $y = a + v + x$, we have $dy = dv + dx$, and thence by substitution the equation becomes

$$adv + adx = adx + vdx + xdx - xdx;$$

or, by erasing the terms that destroy each other, we have $\frac{adv}{v} = dx$, whose integral is $x = a\log cv$; or, since

$$v = y - a - x,$$

we shall have $x = a \log c\,(y - a - x)$. (See Vince's "Fluxions," p. 181.)

(10.) We will now show that if we have a differential equation of $Mdx + Ndy = 0$, of the first order, between two variables x and y, in which the condition $\frac{dM}{dy} = \frac{dN}{dx}$ of integrability is not satisfied, that the condition may still be satisfied after it has been multiplied by a suitable factor; and of course the integral can be found.

For since $Mdx + Ndy = 0$ is not considered as being immediately integrable, it may be supposed to have been obtained by eliminating a constant from an equation of the form $F(x, y) = 0$ and its first differential. Hence, if C stands for the constant, by solving the equation with reference to C, we shall obtain an equation of the form $C = f(x, y)$; consequently, by taking the differential of this, we shall, without reduction, get the differential equation

$$M'dx + N'dy = 0,$$

in which $\frac{dy}{dx}$ or $\frac{dx}{dy}$ must clearly be the same as in

$$Mdx + Ndy = 0,$$

since the two equations result from the elimination of the constant C, from the equation $F(x, y) = 0$ in two different ways; the proposed equation resulting from the elimination of C from $F(x, y) = 0$ by means of its differential equation, and the equation $M'dx + N'dy = 0$ resulting from the immediate differentiation of the equation $C = f(x, y)$.

Hence, eliminating $\frac{dy}{dx}$ from the preceding equations, we shall get $\quad \frac{dy}{dx} = -\frac{M}{N}$ and $\frac{dy}{dx} = -\frac{M'}{N'}$;

consequently, we get $\frac{M}{N} = \frac{M'}{N'}$ such, that M′ and N′ must clearly be like multiples of M and N.

REMARKS.—1. Having found M′ and N′, it is manifest that the integral of $M'dx + N'dy = 0$ will give $C = f(x, y)$, in which C represents the arbitrary constant, and which represents nearly a transformation of the equation $F(x, y) = 0$.

2. Since $M'dx + N'dy = 0$ is an exact differential, it follows, from Euler's Criterion of Integrability (see p. 440), that we shall have

$$\frac{dM'}{dy} = \frac{dN'}{dx}.$$

Hence, if z represents the factor of M and N, which gives $Mz = M'$ and $Nz = N'$, the condition of integrability becomes

$$\frac{d\overline{Mz}}{dy} = \frac{d\overline{Nz}}{dx},$$

which gives

$$(Mdz + zdM) \div dy = (Ndz + zdN) \div dx,$$

or

$$z\left(\frac{dM}{dy} - \frac{dN}{dx}\right) = N\frac{dz}{dx} - M\frac{dz}{dy},$$

which z must satisfy. Having found $Mzdx + Nzdy = du$, it is manifest that the members of this multiplied by any function of u will also be an exact differential; consequently, *there will be an unlimited number of factors that will make the proposed differential an exact differential.*

EXAMPLES.

1. To find the factor which will reduce $ydx - xdy = 0$ to an exact differential.

Here we have $M = y$ and $N = -x$, and thence

$$z\left(\frac{dM}{dy} - \frac{dN}{dx}\right) = N\frac{dz}{dx} - M\frac{dz}{dy},$$

which can clearly be satisfied by putting $z = \frac{1}{y^2}$, and gives $\frac{2}{y^2} = \frac{2}{y^2}$, an identical equation; or, by writing the form $\frac{dx}{x} - \frac{dy}{y} = 0$, and integrating

$$\frac{ydx - xdy}{y^2} = 0, \quad \text{or the form} \quad \frac{dx}{x} - \frac{dy}{y},$$

we have
$$\frac{x}{y} = \text{C} = \text{const.}$$

REMARKS.— $\frac{x}{y}\, d\,\frac{x}{y} = 0$, $\frac{x^2}{y^2}\, d\,\frac{x}{y} = 0$, and, generally,

$$\phi\left(\frac{x}{y}\right) d\,\frac{x}{y} = 0,$$

are also exact differentials of the proposed equation, agreeably to what has been done.

2. To find the factor that reduces $ydx - mxdy = 0$ to an integrable form.

Here, as in the preceding example, we get

$$z = \frac{1}{y^{m+1}}, \quad \text{and thence} \quad \frac{ydx - mxdy}{y^{m+1}} = d\,\frac{x}{y^m}$$

is the transformed differential, whose integral is

$$\frac{x}{y^m} = \text{C} = \text{const.}$$

3. To find the factor that makes $dy + \text{P}ydx = \text{Q}dx$ integrable.

Here $\text{M} = \text{P}y - \text{Q}$ and $\text{N} = 1$, and thence we have

$$\frac{d\text{M}}{dy} - \frac{d\text{N}}{dx} = \text{P} \quad \text{and} \quad z\text{P} = \text{N}\,\frac{dz}{dx} = \frac{dz}{dx},$$

supposing z to be independent of y; consequently, we have $\frac{dz}{z} = \text{P}dx$, whose integral is $\log z = \int \text{P}dx$, supposing the constant to be included under the sign of integration $\int$.

Multiplying the proposed equation by $z = e^{\int Pdx}$, which gives $\log z = \int Pdx$, we have

$$e^{\int Pdx}dy + ye^{\int Pdx}Pdx = e^{\int Pdx}Qdx,$$

whose integral is $$e^{\int Pdx}y = \int e^{\int Pdx}Qdx,$$

and thence $$y = e^{-\int Pdx}\left(\int e^{\int Pdx}Qdx\right),$$

supposing the constant to be indicated by the preceding sign of integration, or the integral may be expressed, as at p. 456,

by $$y = e^{-\int Pdx}\left(\int e^{\int Pdx}Qdx + C'\right).$$

4. To find the factor that makes

$$x^3dy + \left(4x^2y - \frac{1}{\sqrt{1-x^2}}\right)dx = 0$$

integrable.

Here $M = 4x^2y - \frac{1}{\sqrt{(1-x^2)}}$ and $N = x^3$,

and thence we shall have

$$\frac{dM}{dy} - \frac{dN}{dx} = 4x^2 - 3x^2 = x^2;$$

consequently, supposing z to be a function of x only, we shall have $$zx^2 = x^3\frac{dz}{dx} \text{ or } \frac{dz}{z} = \frac{dx}{x},$$

which is clearly satisfied by putting $z = x$. Hence, multiplying the proposed equation by x, we have

$$x^4dy + 4x^3ydx - \frac{xdx}{\sqrt{(1-x^2)}} = 0,$$

whose integral is $x^4y + \sqrt{(1-x^2)} = C$.

5. To find the factor that will make

$$aydy + (cx - by^2)dx = 0$$

an exact differential.

Here $\qquad M = cx - by^2 \quad \text{and} \quad N = ay,$

and thence $\qquad \frac{dM}{dy} - \frac{dN}{dx} = -2by,$

which gives $\qquad -z \times 2by = ay\frac{dz}{dx},$

by supposing z to be independent of y; which gives

$$\frac{dz}{z} = -\frac{2bdx}{a} \quad \text{or} \quad z = e^{-\frac{2bx}{a}}$$

for the sought factor. Hence the transformed differential becomes

$$[aydy + (cx - by^2)\,dx]\,e^{-\frac{2bx}{a}} = 0;$$

whose integral, sometimes called the *primitive*, is

$$\left(by^2 - cx - \frac{ac}{2b}\right)e^{-\frac{2bx}{a}} = C.$$

(See Young, p. 210, &c.)

(11.) We now propose to show how to integrate any homogeneous differential equation consisting of any number of variables.

Thus, let $\quad Mdx + Ndy + Pdz + \&c. = 0$

be a homogeneous differential equation, consisting of any number of variables; then, if the equation is not integrable, it is clear from what is shown at p. 445, that it must be on account of the omission of a homogeneous factor, common to its terms. Hence, if u stands for the omitted factor, we shall have

$$uMdx + uNdy + uPdz + \&c. = du' = 0,$$

the differential being exact. If n denotes the degree of homogeneity of u', we have, from what is shown at pp. 445 and 446, $\quad uMx + uNy + uPz + \&c. = nu';$

consequently, dividing the members of

$$u\mathrm{M}dx + u\mathrm{N}dy + \&\text{c.} = du'$$

by the members of the preceding equation, we shall have

$$\frac{\mathrm{M}dx + \mathrm{N}dy + \&\text{c.}}{\mathrm{M}x + \mathrm{N}y + \&\text{c.}} = \frac{du'}{nu'};$$

consequently, since the right member of this is an exact differential (its integral being $\frac{1}{n}\log u'$), it is plain that

$$\frac{\mathrm{M}dx + \mathrm{N}dy + \&\text{c.}}{\mathrm{M}x + \mathrm{N}y + \&\text{c.}}$$

must also be an exact differential.

It hence follows, *that the factor which makes the proposed differential*

$\mathrm{M}dx + \mathrm{N}dy + \&\text{c.} = 0$ *exact, is* $\frac{1}{\mathrm{M}x + \mathrm{N}y + \&\text{c.}}$;

and thence, if $\mathrm{M}dx + \mathrm{N}dy = 0$ *is the proposed equation, the requisite factor is* $\frac{1}{\mathrm{M}x + \mathrm{N}y}$.

REMARKS.—1. It is clear, from pp. 445 and 446, that the degree of homogeneity of $\mathrm{M}x + \mathrm{N}y +$, &c., when the preceding process is applicable, must be different from naught; and $\mathrm{M}x + \mathrm{N}y +$, &c., must also be different from naught.

2. If $\mathrm{M}dx + \mathrm{N}dy = 0$, and, at the same time, $\mathrm{M}x + \mathrm{N}y = 0$, then, eliminating N from the first of these by means of the second, we shall have

$$\mathrm{M}dx - \frac{\mathrm{M}x}{y}dy = \mathrm{M}y\left(\frac{ydx - xdy}{y^2}\right) = \mathrm{M}yd\frac{x}{y} = 0,$$

which shows that if $\mathrm{M}y$ is a function of $\frac{x}{y}$, the integral can be immediately found in its most general form.

EXAMPLES.

1. To find the factor that makes

$$(xy - y^2)\,dx + (yx + x^2)\,dy = 0,$$

an exact differential.

Since $$Mx + Ny = 2x^2 y,$$

by dividing the given equation by $x^2 y$, we have

$$\left(\frac{1}{x} - \frac{y}{x^2}\right) dx + \left(\frac{1}{x} + \frac{1}{y}\right) dy = d \log xy + d\,\frac{y}{x} = 0,$$

whose integral is $$\log xy + \frac{y}{x} = C.$$

2. To integrate $(x^2 - y^2)\,dx + (xy + x^2)\,dy = 0.$

Here $$Mx + Ny = x^2 (x + y),$$

and thence, dividing the given equation by this, we have

$$\left(\frac{1}{x} - \frac{y}{x^2}\right) dx + \frac{dy}{x},$$

whose integral is $$\log x + \frac{y}{x} = C.$$

3. To integrate $ydx - xdy = 0.$

Here $M = y$ and $N = -x$, and thence $Mx + Ny = 0$; consequently, from what is shown above, we shall have

$$Myd\,\frac{x}{y} = 0, \quad \text{or} \quad y^2 d\,\frac{x}{y} = 0;$$

and this multiplied by $\dfrac{1}{y^2}\,\phi\left(\dfrac{x}{y}\right)$ becomes $\phi\left(\dfrac{x}{y}\right) d\,\dfrac{x}{y} = 0,$

an integrable form, since $\phi\left(\dfrac{x}{y}\right)$ represents a function of $\dfrac{x}{y}$.

4. To integrate $(x^2y - y^2x)\,dx + yx^2dy = 0$.

From $Mx + Ny = x^3y$, we have $\left(\frac{1}{x} - \frac{y}{x^2}\right) dx + \frac{dy}{x}$;

whose integral is $\log x + \frac{y}{x} = C$,

the same as in example 2.

5. To integrate $(x^2y + y^3)\,dx - (x^3 + xy^2)\,dy = 0$.

Here $M = y\,(x^2 + y^2)$ and $N = -\,x\,(x^2 + y^2)$,

and thence we have $Mx + Ny = 0$;

consequently, from what has been shown, we shall have

$$My d\,\frac{x}{y} = y^2\,(x^2 + y^2)\,d\,\frac{x}{y} = 0,$$

and it is easy to perceive that $\frac{1}{y^4}\,\phi\left(\frac{x}{y}\right)$ is the factor, which makes this integrable, since it reduces it to

$$\left(\frac{x^2}{y^2} + 1\right)\phi\left(\frac{x}{y}\right) d\,\frac{x}{y} = F\left(\frac{x}{y}\right) d\,\frac{x}{y} = 0,$$

which is clearly an integrable form, since $F\left(\frac{x}{y}\right)$ is supposed to be a function of $\frac{x}{y}$, at the same time that $\phi\left(\frac{x}{y}\right)$ also denotes a function of $\frac{x}{y}$.

SECTION VII.

INTEGRATION OF DIFFERENTIAL EQUATIONS OF THE FIRST ORDER AND HIGHER DEGREES, AND THE SINGULAR SOLUTIONS OF DIFFERENTIAL EQUATIONS, ETC., BETWEEN TWO VARIABLES.

(1.) It is sometimes said by authors, that differential equations of the first order and higher degrees can not result from the immediate differentiation of any integral, but must arise from the elimination of an integral power of a constant from the integral, by means of its differential equation. (See p. 311.) That what is here affirmed is not universally true, may be proved from the simplest considerations. For (see p. 191) in finding *multiple points of the first kind*, we differentiate the equation of the curve by regarding the co-ordinates at the points of intersection as being independent variables. Thus, in finding the multiple points of the curve whose equation (see p. 191), is

$$ay^2 + cxy - bx^3 = 0,$$

by proceeding, as directed, we have found the differential equation $\qquad 2ady^2 + 2cdxdy - 6bxdx^2 = 0\,;$

which, divided by $2dx^2$, and representing $\dfrac{dy}{dx}$ by p, becomes

$$\frac{dy^2}{dx^2} + \frac{c}{a}\frac{dy}{dx} - \frac{3bx}{a} = p^2 + \frac{c}{a}p - \frac{3bx}{a} = 0,$$

in which $\dfrac{dy}{dx}$ or p is taken on the supposition that, after the differentiation, dy is a function of dx, or contaius it. To find the integral of the preceding equation, we must, by a reverse process, reduce it back to

$$2ady^2 + 2cdxdy - 6bxdx^2 = 0,$$

whose first integral is

$$2aydy + cxdy + cydx - 3bx^2dx = 0;$$

and then the integral of this is

$$ay^2 + cxy - bx^3 = 0,$$

the proposed equation, as it clearly ought to be. Solving the equation

$$p^2 + \frac{c}{a}p - \frac{3bx}{a} = 0, \quad \text{we get} \quad p = -\frac{c}{2a} \pm \frac{1}{2a}\sqrt{c^2 + 12abx};$$

or, since $p = \frac{dy}{dx}$, we have

$$dy = -\frac{cdx}{2a} \pm \frac{d}{2a}\sqrt{(c^2 + 12abx)},$$

whose integral is $y = -\frac{cx}{2a} \pm \frac{(c^2 + 12abx)^{\frac{3}{2}}}{36a^2b} + \text{const.}$

Hence $$\left(y + \frac{cx}{2a}\right)^2 = \frac{(c^2 + 12abx)^3}{(36a^2b)^2},$$

by omitting the constant, or

$$y^2 + \frac{cxy}{a} + \frac{c^2x^2}{4a^2} = \frac{c^6 + 36\,c^4abx + 36^2 \times 12^2a^2b^2x^2 + 12^3a^3b^3x^3}{3^2 \times 12^2a^4b^2},$$

which clearly can not be reduced to the integral

$$ay^2 + cxy - bx^3 = 0,$$

or the proposed equation. If we integrate the equation $dy^2 - a^2dx^2 = 0$, supposing x and y to commence together, by either of the preceding methods, they will be found to give $y^2 = a^2x^2$; while $dy^2 - axdx^2 = 0$, integrated by the first method, gives $y^2 = \frac{ax^3}{3}$, and integrated by the second method, gives $y^2 = \frac{4}{9}ax^3$, which does not agree with the preceding integral.

(2.) The common method of finding the integrals of equations of the form

$$dy^n + Pdy^{n-1}dx + Qdy^{n-2}dx^2 + \quad \ldots\ldots \quad + Udx^n = 0,$$

or its equivalent

$$\left(\frac{dy}{dx}\right)^n + P\left(\frac{dy}{dx}\right)^{n-1} + Q\left(\frac{dy}{dx}\right)^{n-2} + \quad \ldots\ldots \quad + U = 0,$$

consists in solving it like an equation of the nth degree, by regarding $\frac{dy}{dx}$ as the unknown quantity, and of course there will result n equations of the forms

$$\frac{dy}{dx} - p = 0, \quad \frac{dy}{dx} - p' = 0, \quad \frac{dy}{dx} - p'' = 0,$$

and so on, to n equations; p, p', p'', &c., being the roots of the equation.

From these equations we get

$$y - \int pdx = 0, \quad y - \int p'dx = 0, \quad y - \int p''dx = 0,$$

and so on. Hence we shall have

$$\left(y - \int pdx\right) \times \left(y - \int p'dx\right) \times \left(y - \int p''dy\right) = 0,$$

which may be taken to represent the integral of the proposed equation; noticing, that each of the factors may be supposed to be corrected by the addition of the same constant.

For the method of integration here proposed, the reader is referred to Lacroix, "Calcul Intégral," p. 279, &c.; Young, "Integral Calculus," p. 224; and Lardner, p. 318.

EXAMPLES.

1. To find the integral of $y\dfrac{dy^2}{dx^2} + 2x\dfrac{dy}{dx} - y = 0$.

Reducing the equation to the form

$$ydy^2 + 2xdydx - ydx^2 = 0,$$

and taking the integral, regarding x and y as independent variables, we have

$$\frac{y^2dy}{2} + 2yxdx + x^2dy - ydx\,x = 0 \quad \text{and} \quad \frac{y^3}{6} + x^2y - y\frac{x^2}{2} = 0,$$

found on the supposition that x and y commence together, and that y in the last term is constant; but, since y is not constant in the last term, it is clear that the equation has not been obtained on the supposition of x and y being independent variables; noticing, if the last term of the equation had been x or any function of it, the proposed equation might have been obtained on the supposition of x and y being independent variables, and of course a doubt as to the true origin of the proposed equation would have been the result.

Hence, solving the equation on the supposition that x and y are not both independent variables, we have

$$\frac{dy}{dx} = \frac{-x + \sqrt{y^2 + x^2}}{y} \quad \text{and} \quad \frac{dy}{dx} = \frac{-x - \sqrt{(y^2 + x^2)}}{y},$$

which may be put under the forms

$$y\frac{dy}{dx} + x - \sqrt{(y^2+x^2)} = 0 \quad \text{and} \quad y\frac{dy}{dx} + x + \sqrt{(y^2+x^2)} = 0;$$

and by taking the product of these factors, we have

$$\left(y\frac{dy}{dx} + x\right)^2 - (y^2 + x^2) = 0, \quad \text{or} \quad \pm\frac{ydy + xdx}{\sqrt{y^2 + x^2}} = dx,$$

whose integral is

$$\pm\sqrt{y^2 + x^2} = x + C, \quad \text{or} \quad y^2 = 2cx + C^2.$$

2. To find the integral of $dy^2 \pm dx^2 = 2dxdy$.

Integrating on the supposition of the independence of x and y, we have

$$\frac{y^2 \pm x^2}{2} = xy + C, \quad \text{or} \quad y^2 \pm x^2 = 2xy + C.$$

Remarks.—If we take $+$ for $\pm$ in the proposed equation, we have

$$dy^2 + dx^2 = 2dxdy, \quad \text{or} \quad dy^2 - 2dydx + dx^2 = 0,$$

or
$$dy - dx = 0,$$

whose integral is $y - x = C$; the same as by the preceding method. If the proposed differential is

$$dy^2 - dx^2 = 2dxdy,$$

it is clear that the integral found on the principles of the independence of x and y, and their dependence, as in the common method of integration, will not agree; consequently, the origin of the proposed differential is doubtful.

3. To integrate $x\dfrac{dy^2}{dx^2} + x - 1 = 0$, or $\dfrac{dy^2}{dx^2} = \dfrac{1}{x} - 1$.

Multiplying by dx^2, and integrating on the supposition that x and y are independent variables, we have

$$dy^2 = \frac{dx^2}{x} - dx^2, \quad \text{and thence} \quad ydy = dx\log x - xdx,$$

which integrated again, gives

$$\frac{y^2}{2} = x\log x - x - \frac{x^2}{2} + \text{const.},$$

or $$y^2 = 2x \log x - 2x - x^2 + C;$$

consequently, the origin of the differential is doubtful.

REMARKS.—Mr. Young, at p. 226 of his work, finds

$$y = \sqrt{x - x^2} - \tan^{-1}\sqrt{\frac{1-x}{x}} + C$$

for the integral; a result very different from the preceding.

(3.) When only one of the variables x or y enters the proposed equation, and the value of the variable in a function of $\frac{dy}{dx} = p$ can be found; or if p can be found in a function of the variable; then, in solving the equation in the common way, the other variable can be found. Thus, having found $x = F(p)$, or $y = f(p)$,

we get $dx = \frac{dF(p)}{dp}dp$, and $dy = \frac{df(p)}{dp}dp$;

consequently, from

$$dy = pdx, \quad \text{or} \quad dx = \frac{dy}{p},$$

we shall get

$$y = \int pdx = \int p\frac{dF(p)}{dp}dp, \text{ or } x = \int\frac{dy}{p} = \int\frac{df(p)}{pdp}dp,$$

whose integrals will determine the value of y or x.

For integrating

$$y = \int p\frac{dF(p)}{dp}dp$$

by parts, we shall have

$$y = pF(p) - \int F(p)\,dp;$$

so that if $F(p) = \frac{1}{p^2+1}$, we shall get

$$y = \frac{p}{p^2+1} - \int\frac{dp}{1+p^2} = \frac{p}{p^2+1} - \tan^{-1}p;$$

which, from $x = \dfrac{1}{p^2+1}$, gives $p = \sqrt{\dfrac{1-x}{x}}$, and thence

$$y = \sqrt{x - x^2} - \tan^{-1}\sqrt{\frac{1-x}{x}} + C,$$

the result quoted from Mr. Young in the preceding example.

If the equation involves such high powers of x or y that it can not readily be solved, we make such a substitution for

$$\frac{dy}{dx} = p, \quad \text{or} \quad \frac{dx}{dy} = \frac{1}{p} = p',$$

as will reduce the degree of the equation, so that x or y may be found by the common methods of solving equations.

Thus, to integrate

$$x^3 + x\frac{dy}{dx} + \frac{dy^3}{dx^3} = 0,$$

we put

$$\frac{dy}{dx} = xz,$$

and thence, by substitution, get

$$x^3 + x^2z + x^3z^3 = 0, \quad \text{or} \quad x = -\frac{z}{1+z^3},$$

which gives

$$dx = -\frac{dz}{1+z^3} + \frac{3z^3dz}{(1+z^3)^2} = -\frac{(1-2z^3)\,dz}{(1-z^3)^2}.$$

From

$$\frac{dy}{dx} = xz,$$

by substituting the value of x, we have

$$\frac{dy}{dx} = -\frac{z^2}{1+z^3}, \quad \text{or} \quad dy = -\frac{z^2}{1+z^3}\,dx,$$

and, substituting the value of dx, thence

$$dy = -\frac{(1-2z^3)\,z^2\,dz}{(1+z^3)^3} = -\frac{3z^2dz}{(1+z^3)^3} + \frac{2z^2dz}{(1+z^3)^2},$$

whose integral gives

$$y = \frac{1}{2(1+z^3)^2} - \frac{1}{3(1+z^3)} + C.$$

Solving this equation by quadratics, regarding $\frac{1}{1+z^3}$ as being the unknown quantity, we shall get $\frac{1}{1+z^3}$ in a function of y, whose reciprocal gives $1 + z^3$ in a function of y, which, diminished by 1, gives z^3, whose cube root gives the value of z. Hence, x is easily found in a function of y, as required; noticing, that we may clearly proceed in like manner in all analogous cases.

(4.) If the equation involves both variables, in such a way as to make its terms homogeneous relatively to the variables, then, putting $y = xz$ in the equation, if n denotes the degree of homogeneity of the equation, its terms will be divisible by x^n; and we shall have an equation in terms of z and p, whose highest power in z will be z^n.

Hence, if the equation can be solved with reference to z, we shall have $z = \mathrm{F}(p)$; or, if the equation can be solved relatively to p, we shall get $p = f(z)$. Since $y = xz$, we have

$$dy = xdz + zdx = xd\mathrm{F}(p) + \mathrm{F}(p)\,dx,$$

or

$$\frac{dy}{dx} = p = x\,\frac{d\mathrm{F}(p)}{dx} + \mathrm{F}(p),$$

which gives

$$\frac{dx}{x} = \frac{d\mathrm{F}(p)}{p - \mathrm{F}(p)},$$

whose integral can be found in a function of p; and thence from

$$y = xz = x\mathrm{F}(p),$$

by eliminating p, we get y in a function of x, as required.

In much the same way, from

$$p = \frac{dy}{dx} = f(z), \quad \text{and from} \quad dy = xdz + zdx,$$

we have $\dfrac{xdz}{dx} + z = f(z)$, or $\dfrac{xdz}{dx} = f(z) - z$,

which gives
$$\frac{dx}{x} = \frac{dz}{f(z) - z},$$

which, integrated, expresses x in a function of z; and thence, from $y = xz$, we express y in a function of z; consequently, eliminating z from the values of x and y, we shall get y in a function of x.

EXAMPLES.

1. Given $y - xp = x\sqrt{1 + p^2}$ to find the integral.

By putting $y = xz$, the equation reduces to

$$z - p = \sqrt{1 + p^2}, \quad \text{or} \quad z = p + \sqrt{1 + p^2},$$

which gives
$$dz = dp + \frac{pdp}{\sqrt{1 + p^2}};$$

and from
$$dy = pdx = xdz + zdx,$$

we have
$$\frac{dx}{x} = \frac{dz}{p - z} = -\frac{dp}{\sqrt{1 + p^2}} - \frac{pdp}{1 + p^2},$$

whose integral clearly gives

$$\log x = -\log(p + \sqrt{1 + p^2}) - \log\sqrt{1 + p^2} + \log C;$$

and thence we have

$$x = \frac{c}{\sqrt{1 + p^2}\,(p + \sqrt{1 + p^2})};$$

and since
$$z = p + \sqrt{1 + p^2},$$

we have
$$xz = y = \frac{c}{\sqrt{1 + p^2}};$$

consequently, we hence get

$$x = \frac{y^2}{c + \sqrt{c^2 - y^2}}$$

for one form of the proposed integral.

Otherwise.—By squaring the members of the proposed equation, we have

$$y^2 - 2xyp + x^2p^2 = x^2 + x^2p^2;$$

or, erasing the terms that destroy each other, we have

$$\frac{y^2 - x^2}{2xy} = p;$$

or, since $y = xz$, we shall have

$$p = \frac{z^2 - 1}{2z} = f(z);$$

and thence $\frac{dx}{x} = \frac{dz}{f(z) - z}$ reduces to $\frac{dx}{x} = -\frac{2zdz}{1 + z^2}$, whose integral is

$$\log x = \log \frac{c}{1 + z^2} \quad \text{or} \quad x = \frac{c'}{1 + z^2},$$

which, since $z = \frac{y}{x}$, is immediately reducible to

$$x(x^2 + y^2) = x^2 c, \quad \text{or} \quad x^2 + y^2 = c'x.$$

REMARKS.—It is easy to show that this integral agrees with the integral found by the first method, since it can be put under the form

$$c + \sqrt{c^2 - y^2} = \frac{y^2}{x}, \quad \text{or} \quad \sqrt{c^2 - y^2} = \frac{y^2}{x} - c,$$

which, by squaring and an obvious reduction, becomes $x^2 + y^2 = 2cx$, which agrees with $x^2 + y^2 = c'x$, when we put c' for $2c$. For the first of these solutions, the reader is referred to p. 229 of Mr. Young's work.

2. To find the integral of $y^2 - px^2 = py^2$.

Putting xz for y, the equation immediately reduces to

$$z^2 - p = pz^2, \quad \text{which gives} \quad p = \frac{z^2}{1+z^2} = f(z).$$

Hence, we have $\dfrac{dx}{x} = \dfrac{dz}{f(z) - z} = -\dfrac{dz\,(1+z^2)}{z^3 - z^2 + z}$.

Since $-\dfrac{1+z^2}{z^3 - z^2 + z} = -\dfrac{1}{z} - \dfrac{1}{z^2 - z + 1}$, we shall have

$$\frac{dx}{x} = -\frac{dz}{z} - \frac{dz}{z^2 - z + 1} = -\frac{dz}{z} - \frac{\frac{3}{4}\,dz}{\frac{3}{4}\left\{\frac{3}{4} + \left(z - \frac{1}{2}\right)^2\right\}},$$

whose integral gives

$$\log x = -\log z - \frac{A}{3 \div 4} = -\log cz - \frac{4A}{3},$$

in which A is an arc of a circle, whose radius $= \dfrac{\sqrt{3}}{2}$ and tangent $= z - \dfrac{1}{2}$. Since $z = \dfrac{y}{x}$, if we put $\dfrac{y}{x}$ for z in this equation, we shall have the required integral.

(5.) Supposing $\dfrac{dy}{dx} = p$, we will now proceed to show how to integrate the equation $y = xp + F(p)$, on the supposition that $F(p)$ is independent of x and y; noticing, that this equation is called *Clairaut's form.*

By differentiating the members of the equation, we have

$$\frac{dy}{dx} = p = p + x\frac{dp}{dx} + \frac{dF(p)}{dp}\frac{dp}{dx},$$

or, erasing the terms that destroy each other, we have

$$\left(x + \frac{dF(p)}{dp}\right)\frac{dp}{dx} = 0.$$

This equation is satisfied by putting $\frac{dp}{dx} = 0$ or $dp = 0$, whose integral is $p = C =$ const., and of course $y = Cx$ is the proposed equation.

The preceding equation can also be satisfied by putting its other factor equal to naught, which gives

$$x + \frac{dF(p)}{dp} = 0;$$

consequently, if $\frac{dF(p)}{dp}$ is a function of p, by finding p from this, and substituting its value in the proposed equation, we shall get an equation between x and y, which does not contain any arbitrary constant, and is hence called the *singular solution* of the proposed equation. Thus, if

$$y = xp + ap^2, \quad \text{we have} \quad y = Cx + aC^2$$

for the integral, or $p = -\frac{x}{2} = 0$ is the singular solution.

Similarly, if $\qquad y = px + a(1 + p^2)$,

we shall have $\qquad y = Cx + a(1 + C^2)$

for the integral, and $-\frac{x}{2a}$ is the singular solution.

REMARKS.—1. If we have the equation $y = Px + Q$, in which P and Q are functions of p, then, by differentiating, we shall get

$$p = \frac{dy}{dx} = P + \frac{xdP}{dx} + \frac{dQ}{dx}, \quad \text{or} \quad dx + x\frac{dP}{P-p} = -\frac{dQ}{P-p}.$$

Taking the integral of this, by the form given in (4) at p. 455, we have

$$x = e^{-\int\frac{dP}{P-p}}\left\{\int e^{\int\frac{dP}{P-p}}\frac{-dQ}{P-p} + C\right\},$$

by changing y and dy into x and dx; then, eliminating p from this and the proposed equation, we shall get the integral between x and y.

2. It is manifest from what has been done in the first part of this section, that in the application of the Differential and Integral Calculus to estimate the changes of position of bodies, which result from the violent and sudden actions of powerful forces, we ought generally to take the differentials of the variables on the supposition that they are independent of each other, since the tendency of the actions of the forces is plainly to introduce multiple points or cusps into the motions of the bodies. Reciprocally, in finding the integrals of differentials thus found, we ought to proceed on the supposition of the independence of the variables, as explained at p. 484, &c.

(6.) From what has been done, we are naturally led to the consideration of what are called the *Singular Solutions of Differential Equations of the First Order.*

1. If $F(x, y, c) = 0$, in which c represents a constant, and we differentiate the equation, regarding c alone as variable, we shall have $\frac{dF(x, y, c)}{dc} dc = 0$;

then, if, as in the example at p. 187, we eliminate c from the equation $F(x, y, c) = 0$ and $\frac{dF(x, y, c)}{dc} = 0$,

when its dimensions exceed the first degree, the result will (generally) be what is called a *singular solution* of the proposed equation.

2. If we regard x and y as being functions of c, then, by differentiating $F(x, y, c) = 0$ with reference to c, we shall have

$$\frac{d\mathrm{F}(x, y, c)}{dx}\frac{dx}{dc}dc + \frac{d\mathrm{F}(x, y, c)}{dy}\frac{dy}{dc}dc + \frac{d\mathrm{F}(x, y, c)}{dc}dc = 0;$$

or, since
$$\frac{d\mathrm{F}(x, y, c)}{dc} = 0,$$
we have

$$\frac{d\mathrm{F}(x, y, c)}{dx}\frac{dx}{dc}dc + \frac{d\mathrm{F}(x, y, c)}{dy}\frac{dy}{dc} \times dc = 0.$$

Because this equation must evidently be satisfied so as to leave
$$\frac{d\mathrm{F}(x, y, c)}{dx}, \quad \text{or} \quad \frac{d\mathrm{F}(x, y, c)}{dy}$$
arbitrary, we must have either

$$\frac{dx}{dc} = 0, \quad \text{or} \quad \frac{dy}{dc} = 0;$$

which may clearly be used instead of

$$\frac{d\mathrm{F}(x, y, c)}{dc} = 0.$$

Similarly, if $p = \frac{dy}{dx}$, and we have the differential equation $f(x, y, p) = 0$ such, that $\mathrm{F}(x, y, c) = 0$ represents its singular solution; then, solving $f(x, y, p) = 0$ relatively to c, we get the form $c = \theta(x, y, p)$, which reduces

$$\mathrm{F}(x, y, c) = 0 \quad \text{to the form} \quad \mathrm{F}(x, y, \theta) = 0,$$

by using θ to stand for the function $\theta(x, y, p)$, or its equivalent c.

If, for brevity, we represent the first member of this equation by u, then, since the function θ may contain x and y, by taking the differential of $u = 0$, we shall have

$$\left(\frac{du}{dx} + \frac{du}{d\theta}\cdot\frac{d\theta}{dp}\cdot\frac{dp}{dx}\right)dx + \left(\frac{du}{dy} + \frac{du}{d\theta}\cdot\frac{d\theta}{dp}\cdot\frac{dp}{dy}\right)dy = 0,$$

which must clearly be satisfied so as to leave dx and dy arbitrary. Hence, we may clearly put the coefficient of dx or dy equal to naught, and shall thence get

$$\frac{du}{dx} + \frac{du}{d\theta} \cdot \frac{d\theta}{dp} \cdot \frac{dp}{dx} = 0, \quad \text{or} \quad \frac{du}{dy} + \frac{du}{d\theta} \cdot \frac{d\theta}{dp} \cdot \frac{dp}{dy} = 0,$$

which give

$$\frac{dp}{dx} = -\frac{du}{dx} \div \frac{du}{d\theta} \cdot \frac{d\theta}{dp}, \quad \text{or} \quad \frac{dp}{dy} = -\frac{du}{dy} \div \frac{du}{d\theta} \cdot \frac{d\theta}{dp}.$$

Because θ is used for $\theta(x, y, p)$, or its equivalent c, it is clear that $\frac{du}{d\theta} = \frac{du}{dc}$, which (by 1), for the singular solution, must equal naught; consequently, because $\frac{du}{d\theta} = 0$, we must have $\frac{dp}{dx} =$ infinity, or $\frac{dp}{dy} =$ infinity, and thence $\frac{dx}{dp} = 0$, or $\frac{dy}{dp} = 0$, *which are the conditions for finding the singular solutions of differential equations of the first order.* Hence, *if* p *is eliminated from the proposed differential equation by either of these conditions, and if the result satisfies the proposed differential equation, it will be the singular solution of it.*

3. To simplify the applications of what has been done as much as possible, we shall represent the proposed differential equation $f(x, y, p) = 0$ by v, which gives, supposing p to be a function of x and y,

$$\frac{dv}{dx} dx + \frac{dv}{dy} dy + \frac{dv}{dp} \cdot \frac{dp}{dx} dx + \frac{dv}{dp} \cdot \frac{dp}{dy} dy = 0,$$

and thence we get

$$\frac{dv}{dp} = -\left(\frac{dv}{dx} + \frac{dv}{dy} \cdot \frac{dy}{dx}\right) \div \left(\frac{dp}{dx} + \frac{dp}{dy} \cdot \frac{dy}{dx}\right),$$

which reduces $\frac{dv}{dp}$ to naught, since $\frac{dp}{dx}$ or $\frac{dp}{dy}$ in the divisor is infinite. Hence, $\frac{dv}{dp} = 0$ gives the values of p, that give the singular solution. (See Young, p. 232, &c.) It may be noticed if $u = \text{F}(x, y, \theta) = 0$ does not contain y, that we must here regard x as being a function of y, regarded as being the independent variable, and put $p' = \frac{dx}{dy}$ for p. (See Young, p. 237.)

EXAMPLES.

1. To find the singular solution of

$$v = (x + y)p - xp^2 - (a + y) = 0.$$

Here $\frac{dv}{dp} = 0$ becomes $x + y - 2xp = 0$, which gives

$$p = \frac{x + y}{2x};$$

which, being put for p in the given equation, gives

$$v = \frac{(x+y)^2}{2x} - \frac{(x+y)^2}{4x} - (a + y) = \frac{(x+y)^2}{4x} - (a + y) = 0.$$

Hence, we easily get

$$x - y = 2\sqrt{ax}, \quad \text{or} \quad y = x - 2\sqrt{ax},$$

which gives $$\frac{dy}{dx} = p = 1 - \frac{a}{\sqrt{ax}}.$$

From $y = x - 2\sqrt{ax}$ we get

$$x + y = 2x - 2\sqrt{ax} \quad \text{and} \quad a + y = a + x - 2\sqrt{ax};$$

and since $p = 1 - \frac{a}{\sqrt{ax}}$, we thence get

$$2(x - \sqrt{ax})\left(1 - \frac{a}{\sqrt{ax}}\right) - x\left(1 - \frac{a}{\sqrt{ax}}\right)^2 - (a + x - \sqrt{ax}) =$$

$$2(\sqrt{x} - \sqrt{a})^2 - 2(\sqrt{x} - \sqrt{a})^2 = 0;$$

consequently, since $y = x - 2\sqrt{ax}$ satisfies the proposed equation, it must be its singular solution.

2. To find the singular solution of

$$v = x^2 + 2xyp + (a^2 - x^2)p^2 = 0.$$

From $\frac{dv}{dp} = 0$ we get $xy + (a^2 - x^2)p = 0$;

which, multiplied by p and subtracted from the proposed equation, gives $x^2 + xyp = 0$ or $p = -\frac{x}{y}$;

which, substituted in the proposed equation, gives

$$x^2 + y^2 - a^2 = 0;$$

consequently, since this satisfies the proposed equation, it must be its singular solution.

3. To find the singular solution of $xp - y = \sqrt{(x^2 + y^2)}$, or, more properly, of $(xp - y)^2 = x^2 + y^2$.

Here $v = x^2p^2 - 2xyp - x^2 = 0,$

gives $\frac{dv}{dp} = 0$, or its equivalent

$$x^2p - xy = 0, \quad \text{or} \quad p = \frac{y}{x};$$

which reduces the equation to

$$-y^2 - x^2 = 0, \quad \text{or} \quad -y^2 = x^2,$$

as required.

4. To find the singular solution of

$$x^2p^2 - 2xyp + y^2 - x^2 - p^2x^2 = 0.$$

Here $\frac{dv}{dp} = 0$ becomes

$$2x^2p - 2xy - 2x^2p = 0, \quad \text{or} \quad -2xy = 0,$$

which clearly shows that the question does not admit of a singular solution.

5. To find the singular solution of

$$v = (x^2 - 2y^2)\,p^2 - 4xyp - x^2 = 0.$$

Here $\dfrac{dv}{dp} = 0$ becomes $(x^2 - 2y^2)\,p - 2xy = 0$,

which, multiplied by p, and the product subtracted from the given equation, gives

$$- 2xyp - x^2 = 0, \quad \text{or} \quad p = -\frac{x}{2y};$$

consequently, substituting this in the proposed equation, we have $(x^2 + 2y^2)\,x^2 = 0$, which evidently gives $x^2 + 2y^2 = 0$ for the singular solution.

6. To find the value of c, which gives the singular solution of

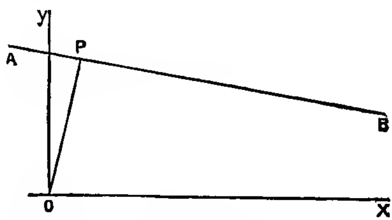
$$y = x + (c - 1)^2 (c - x)^2.$$

Differentiating the equation by regarding c alone as variable, we have

$$2\,(c - 1)\,(c - x)^2 + 2\,(c - 1)^2\,(c - x) = 0, \quad \text{or} \quad c - x + c - 1 = 0,$$

which gives $c = \dfrac{x+1}{2}$ for the particular solution, since the values $c = 1$ and $c = x$, which also satisfy the differential equation, clearly correspond to particular integrals.

7. To find a curve such, that the perpendiculars drawn



from a given point on its tangents shall be constant, or equal to each other.

Let O be the given point, taken for the origin of the co-ordinates, and OP for the perpendicular from the given point or origin to the given line AB; then, we may clearly suppose

$$Y - y = \frac{dy}{dx}(X - x), \quad \text{or its equivalent} \quad Y = pX + y - px,$$

to be the equation of AB, regarded as touching a circle having OP $=$ R for its radius, center O, and $Ox = x$, $Oy = y$, for the rectangular co-ordinates of the point of contact P of the tangent and circle.

Supposing y to decrease when x increases, we shall clearly have $-\frac{dy}{dx} = -p$ for the tangent of the angle yOP, and $y - px$ equals the part of the axis of y between AB and the origin O. Because $\sqrt{p^2 + 1}$ equals the secant of the angle yOP, it is manifest from known principles of trigonometry that we shall have

$$\frac{y - px}{\sqrt{p^2 + 1}} = \text{R} = \text{const.}$$

for the invariable expression of the perpendicular from the origin to the tangents to the sought curve, which must hence clearly be a circle, having O for its center and OP $=$ R for its radius. It is manifest that the preceding equation may be written in the form

$$y = px + \text{R}\sqrt{p^2 + 1},$$

an equation that agrees with Clairaut's form of differential equations given at p. 494, whose integral is there shown to be

$$y = cx + \text{R}\sqrt{c^2 + 1}.$$

The singular solution of this gives

$$c = \frac{-x}{\sqrt{(\text{R}^2 - x^2)}},$$

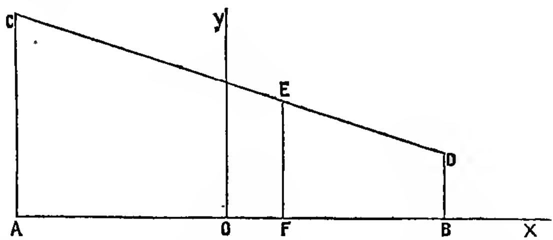
which, being put for c, we have

$$y = \frac{x^2}{\sqrt{(R^2 - x^2)}} + \frac{R^2}{\sqrt{(R^2 - x^2)}} = \sqrt{(R^2 - x^2)},$$

and thence $$x^2 + y^2 = R^2$$

is the singular solution of the proposed question.

8. To find a curve such, that the product of the two perpendiculars from two given points on any tangent shall be constant or invariable.



Let A and B represent the given points through which the axis of x is supposed to pass, the origin of the co-ordinates being at O, the middle point between A and B; then we may suppose that the sought curve touches one of the tangents CD at the point E to the right of Oy, the axis of y, and that OF and FE represent the x and y which correspond to the point of contact of the curve and tangent.

Representing AO = OB by a we shall have

$$AF = a + x \quad \text{and} \quad FB = a - x;$$

then, as in the preceding question, supposing y to decrease when x increases, we shall have $-p =$ the tangent of the angle of inclination of CD to the axis of x, and thence

$$AC = y - (a + x)p \quad \text{and } BD = y + (a - x)p;$$

consequently, by dividing these by $\sqrt{(p^2 + 1)}$, we, as in the preceding example, get

$$\frac{y - (a + x)p}{\sqrt{(p^2 + 1)}} \quad \text{and} \quad \frac{y + (a - x)p}{\sqrt{(p^2 + 1)}}$$

for the perpendiculars from the points A and B to the tangent CD; and thence, if b^2 denotes their product, we shall have

$$\frac{y - (a + x)p}{\sqrt{(p^2 + 1)}} \times \frac{y + (a - x)p}{\sqrt{(p^2 + 1)}} = b^2;$$

or, by performing the indicated multiplication, we have

$$\frac{y^2 - 2pxy - p^2(a^2 - x^2)}{p^2 + 1} = b^2$$

for one form of the equation of the sought curve.

Solving the equation for y, we readily get

$$y = px \pm \sqrt{(b^2 + m^2p^2)},$$

in which $m^2 = a^2 + b^2$; this being a differential equation of Clairaut's form, we shall, as heretofore, get

$$y = Cx \pm \sqrt{b^2 + m^2C^2}$$

for its integral, C being the arbitrary constant.

By taking the differential of this equation, supposing C alone to be variable, we readily get

$$C = \frac{\mp bx}{m\sqrt{(m^2 - x^2)}}$$

for the singular solution. Taking $-bx$ for $\pm bx$ in C, and using $+$ for $\pm$ in y, we get from what has been done

$$m^2y^2 + b^2x^2 = m^2b^2$$

for the equation of the sought curve, when the perpendiculars to the tangents do not fall on opposite sides of the tangents, and it is manifest that the curve is an ellipse;

noticing, if the perpendiculars are drawn on opposite sides of the tangents, that the singular solution will clearly be an hyperbola.

9. To find a curve such, that the length of the normal shall be a (given) function of the distance of its foot from the origin of the abscissas. (See Lacroix, p. 466.)

Supposing x and y to represent the co-ordinates of any point of the proposed curve, it is easy to perceive that

$$y\sqrt{1+\frac{dy^2}{dx^2}} \quad \text{and} \quad x+y\frac{dy}{dx}$$

represent the lengths of the normal and the distance of its foot from the origin of the co-ordinates; consequently,

$$y\sqrt{1+\frac{dy^2}{dx^2}}=f\left(x+y\frac{dy}{dx}\right)$$

will express the differential equation of the question.

It is easy to perceive that the equation $(x-a)^2+y^2=C$, in which C is the arbitrary constant, by putting $C=f(a)^2$ will satisfy the question, and be the complete integral, since it contains the arbitrary constant C. For by taking the differential of

$$(x-a)^2+y^2=C,$$

we have

$$(x-a)\,dx+ydy=0,$$

which gives $a=x+y\dfrac{dy}{dx}$, and $x-a=-y\dfrac{dy}{dx}$,

so that

$$y^2\frac{dy^2}{dx^2}+y^2=f(a)^2=f\left(x+y\frac{dy}{dx}\right)^2;$$

and taking the square root of the members of this, we have

$$y\sqrt{1+\frac{dy^2}{dx^2}}=f\left(x+y\frac{dy}{dx}\right),$$

agreeing with the assumed differential equation. It is manifest that $(x-a)^2+y^2=c$

is the equation of a circle, the axis of x passing through its center, a being the abscissa of its center, and

$$c=f(a)^2$$

is the square of its radius.

If we take the differential of

$$(x-a)^2+y^2=c=f(a)^2,$$

regarding a alone as variable, we shall have

$$-(x-a)=\frac{dc}{2da}=f(a)f'(a);$$

then, by eliminating a from

$$(x-a)^2+y^2=f(a)^2 \quad \text{and} \quad -(x-a)=f(a)f'(a),$$

the result will be the singular solution (called by Lacroix, the particular solution) of the proposed differential equation.

REMARKS.—1st. If we put

$$c=ka \quad \text{in} \quad (x-a)^2+y^2=c,$$

it will become $(x-a)^2+y^2=ka,$

whose differential being taken by regarding a alone as variable, is $-(x-a)=\frac{k}{2},$

which gives $a=x+\frac{k}{2},$

and thence the equation $(x-a)^2+y^2=ka$ reduces to

$$\frac{k^2}{4}+y^2=k\left(x+\frac{k}{2}\right), \quad \text{or} \quad y^2=k\left(x+\frac{k}{4}\right),$$

the equation of a parabola, the origin of the co-ordinates being at the focus of the parabola; noticing, that this is a singular solution, comprehended under the general singular solution given above.

2d. The equations

$$(x-a)^2+y^2=f(a)^2 \quad \text{and} \quad -(x-a)=f(a)f'(a),$$

when a is eliminated, or supposed to be eliminated, from them, give, when taken together, a result which is sometimes called the general integral of the differential equation

$$y\sqrt{1+\frac{dy^2}{dx^2}}=f\left(x+y\frac{dy}{dx}\right) \quad \text{(see p. 505)},$$

while $\quad (x-a)^2+y^2=f(a)^2,$

which involves the arbitrary function $f(a)^2$, is called the complete integral of the same equation. Thus,

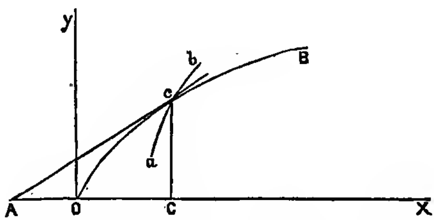
$$xz^2-yz+a=0 \quad \text{and} \quad 2xz-y=0,$$

given at p. 187, may be supposed to correspond to

$$(x-a)^2+y^2=f(a)^2 \quad \text{and} \quad -(x-a)=f(a)f'(a);$$

and $y^2=4ax$, at p. 187, resulting from the elimination of z, corresponds to the elimination of a from the preceding equations.

10. To find the equation of a curve which cuts a curve having a variable parameter, at any proposed angle.



Let OB represent the proposed curve when referred to its rectangular axes as in the figure, and *acb* the cutting curve, when referred to the same axes; then, if

$$\frac{dy}{dx} = p' \quad \text{and} \quad \frac{dy}{dx} = p$$

in the proposed and sought curves, stand for the tangents which the tangents to their arcs at their point *c* of intersection make with the axis of x, we shall, from a well-known formula of trigonometry, get

$$\tan\phi = \frac{\dfrac{dy}{dx} - p'}{1 + p'\dfrac{dy}{dx}}$$

for the tangent of the angle which the sought curve makes with the proposed curve at their common point of intersection, which is supposed to be a given angle; consequently, representing $\tan\phi$ by a, we get

$$a = \frac{\dfrac{dy}{dx} - p'}{1 + \dfrac{dy}{dx}p'}, \quad \text{and thence have} \quad \frac{dy}{dx} = \frac{p' + a}{1 - ap'};$$

and if ϕ is 90°, or a the tangent of a right angle, it becomes infinite, and the preceding equation reduces to $\dfrac{dy}{dx} = -\dfrac{1}{p'}$.

Thus, if $y = bx$ is the equation of the proposed curve, it gives

$$b = p' = \frac{y}{x}, \quad \text{and thence} \quad \frac{dy}{dx} = \frac{\dfrac{y}{x} + a}{1 - a\dfrac{y}{x}},$$

or its equivalent $\quad (x - ay)\,dy - (y + ax)\,dx = 0.$

Because this equation is reducible to the form

$$\frac{d\left(\frac{y}{x}\right)}{1+\left(\frac{y}{x}\right)^2} = a\,\frac{xdx + ydy}{x^2 + y^2},$$

by taking the integral, and using a log C for the arbitrary constant, we have

$$\tan^{-1}\frac{y}{x} = a \log C \sqrt{x^2 + y^2}.$$

Because log represents the hyperbolic logarithm, if A stands for the base of a system of logarithms whose modulus is a, by writing Log for a log, by the nature of logarithms, the preceding equation may be written in the form

$$\text{Log } A^{\tan^{-1}\frac{y}{x}} = \log C \sqrt{x^2 + y^2},$$

in which Log denotes a logarithm taken in the system whose base is A. Putting

$$\tan^{-1}\frac{y}{x} = \theta \quad \text{and} \quad \sqrt{x^2 + y^2} = r,$$

by returning from the logarithms to their numbers, we shall get $A^\theta = Cr$ for the equation of the sought curve. If we suppose $r = 1$ when $\theta = 0$, the preceding equation gives $C = 1$, and thence the preceding equation is reduced to $A^\theta = r$, which is clearly the logarithmic spiral, ϕ being the constant angle at which the radius vector r cuts it, and A the base of the system of logarithms, represented by it, $a = \tan\phi$ being the modulus. (See p. 136.)

Remarks.—1. If $\phi = 90°$, $a = \tan\phi =$ infinity.

2. Hence the equation

$$\frac{d\,\frac{y}{x}}{1+\left(\frac{y}{x}\right)^2} = a\,\frac{xdx + ydy}{2\,(x^2 + y^2)}$$

clearly shows that we must have $xdx + ydy = 0$. By taking the integral of this, we clearly get $x^2 + y^2 = C$, which evidently shows the sought curve to be a circle, C being the square of its radius.

For another example, we will show how to find the curve which cuts a series of parabolas whose general equation is $y^n = ax^m$ at right angles.

From what is shown at p. 509, we must take $\frac{dy}{dx} = -\frac{1}{p'}$, and substitute in it the value of $p' = \frac{dy}{dx}$, as determined from the equation of the proposed curve, and then eliminate from the result the a as found from the equation of the proposed curve.

Thus, we have $$\frac{dy}{dx} = \frac{m}{n}\, a\, \frac{x^{m-1}}{y^{n-1}},$$

which, put for p', reduces

$$\frac{dy}{dx} = -\frac{1}{p'} \quad \text{to} \quad \frac{dy}{dx} = -\frac{n}{m}\,\frac{y^{n-1}}{ax^{m-1}};$$

whose members, multiplied by the corresponding members of the given equation, we get

$$\frac{y^n dy}{dx} = -\frac{n}{m}\,\frac{y^{n-1}x^m}{x^{m-1}},$$

or its equivalent $\quad mydy + nxdx = 0;$

whose integral is clearly the ellipse

$$my^2 + nx^2 = C,$$

in which the constant C represents the product of the squares of the semi-axes of the ellipse.

SECTION VIII.

INTEGRATION OF DIFFERENTIAL EQUATIONS OF THE SECOND AND HIGHER ORDERS, BETWEEN TWO VARIABLES.

(1.) The most general form of a differential equation of the second order between two variables, may evidently be supposed to be reduced so as to contain $x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$, together with constants.

(2.) Supposing $u = 0$ to be the primitive or integral of the second order of the preceding differential equation, containing the two arbitrary constants, C and C′, which have clearly resulted from two successive integrations of the proposed equation, then, by taking the first and second differentials of $u = 0$, regarding x as being the independent variable, we shall have the three equations $u = 0$, $du = 0$, $d^2 u = 0$; consequently, eliminating C and C′ from these equations, we shall clearly obtain the proposed equation of the second order between $x, y, \frac{dy}{dx}, \frac{d^2 y}{dx^2}$, which is clearly independent of the constants C and C′. Now, if we eliminate C′ from $u = 0$ and $du = 0$, we shall get a differential equation of the first order denoted by $V = 0$, involving the constant C; and, in like manner, by eliminating C from $u = 0$ and $du = 0$, we shall get an equation of the first order denoted by $V' = 0$, involving the constant C′. It is easy to perceive that if we eliminate C between $V = 0$ and $dV = 0$, or eliminate C′ between $V' = 0$ and $dV' = 0$, we shall obtain the proposed differential of the second order, so that $V = 0$ and $V' = 0$ will each be a differential equation of the first order of the proposed equation of the second order. Hence, *if we eliminate* $\frac{dy}{dx}$ *from* $V = 0$ *and* $V' = 0$, *it is clear that the result will be* $u = 0$, *the primitive of the proposed differential equation of the second order.* (See Lacroix, pp. 292 and 293.)

Thus, supposing $y + ax + b = 0$ to represent the primitive (or second) integral of a differential equation of the second order, having a and b for its arbitrary constants, then, by differentiating it, we get $\frac{dy}{dx} + a = 0$, which is said to be obtained from the primitive by a *direct differentiation.* But if we divide the primitive by x, we get $\frac{y+b}{x} + a = 0$, whose differential gives

$$xdy - (y + b)\,dx = 0, \quad \text{or} \quad y - x\frac{dy}{dx} + b = 0,$$

which is said to be obtained from the primitive by *an indirect differentiation.*

Thus we have obtained the differential equations

$$\frac{dy}{dx} + a = 0 \quad \text{and} \quad y - x\frac{dy}{dx} + b = 0,$$

which are both of the first order, the first containing the constant a, and the second the constant b. If we differentiate these equations, they will concur in giving $\frac{d^2y}{dx^2} = 0$ for the differential equation of the second order, of which the two preceding equations will be the first integrals; and if we eliminate $\frac{dy}{dx}$ from

$\frac{dy}{dx} + a = 0$ and $y - x\frac{dy}{dx} + b = 0$, we get $y + ax + b = 0$, which is the primitive or second integral of $\frac{d^2y}{dx^2} = 0$.

In like manner, it is clear that a differential equation of the third order has three differential equations of the second order for its first integrals, &c.; and so on, to any extent.

(3.) We now propose to show that any differential equation between two variables, has an integral, such, that the nth order, in its most general form, contains n arbitrary constants.

For conceiving the equation to be solved with respect to the differential coefficient of the nth order, we shall clearly get $\frac{d^ny}{dx^n}$ = a function of

$$x,\quad y,\quad \frac{dy}{dx},\quad \frac{d^2y}{dx^2},\quad \ldots\ldots \frac{d^{n-1}y}{dx^{n-1}};$$

which, by successive differentiations, gives

$$\frac{d^{n+3}y}{dx^{n+3}} = \text{ a function of } x,\quad y,\quad \frac{dy}{dx},\quad \frac{d^2y}{dx^2},\quad \ldots\ldots \frac{d^ny}{dx^n};$$

$$\frac{d^{n+2}y}{dx^{n+2}} = \text{ a function of } x,\quad y,\quad \frac{dy}{dx},\quad \frac{d^2y}{dx^2},\quad \ldots\ldots \frac{d^{n+1}y}{dx^{n+1}},$$

and so on, to any required extent. It is hence easy to perceive that, by obvious substitutions,

$$\frac{d^ny}{dx^n},\quad \frac{d^{n+1}y}{dx^{n+1}},\quad \frac{d^{n+2}y}{dx^{n+2}},\quad \text{\&c.},$$

are all known functions of

$$x,\quad y,\quad \frac{dy}{dx},\quad \frac{d^2y}{dx^2},\quad \ldots\ldots\ldots \frac{d^{n-1}y}{dx^{n-1}},$$

which clearly (generally) consists of n terms.

From Taylor's theorem (see p. 15), we shall have

$$y' = y + \frac{dy}{dx} h + \frac{d^2y}{dx^2} \frac{h^2}{1.2} + \frac{d^3y}{dx^3} \frac{h^3}{1.2.3} +, \&c.;$$

consequently, if x_1 is such a value of x as does not make

$$y, \quad \frac{dy}{dx}, \quad \frac{d^2y}{dx^2}, \quad \ldots\ldots \quad \frac{d^{n-1}y}{dx^{n-1}},$$

in the preceding differential equation, any of them, infinite; then, supposing x_1 to be substituted for x in

$$y, \quad \frac{dy}{dx}, \quad \frac{d^2y}{dx^2}, \quad \ldots\ldots \quad \frac{d^{n-1}y}{dx^{n-1}},$$

and that A, A_1, A_2, ... A_{n-1}, represent their resulting values, if we change h into $x - x_1$ in the preceding expansion, and change y' into y after the substitutions of the values of $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots\ldots \frac{d^{n-1}y}{dx^{n-1}}$,
we shall have

$$y = A + A'(x - x_1) + A_2 \frac{(x - x_1)^2}{1.2} +$$

$$\ldots\ldots \frac{A_{n-1}(x - x_1)^{n-1}}{1.2\ldots\ldots(n-1)} + \frac{d^ny}{dx^n} \frac{(x - x_1)^n}{1.2\ldots\ldots n} +, \&c.,$$

for the integral of the proposed differential equation, whose first n terms clearly contain A, A_1, &c., for the n arbitrary constants required.

(4.) Any differential equation of the nth order between x and y must have n differentials of the order $n - 1$ for its first integrals.

For, as in the preceding proposition, Taylor's theorem gives $y' = y + \frac{dy}{dx} h + \frac{d^2y}{dx^2} \frac{h^2}{1.2} + \frac{d^3y}{dx^3} \frac{h^3}{1.2.3} +, \&c.;$

which, by changing h into $- x$, and representing the value of y', corresponding to $x = 0$ by A, becomes

$$A = y - \frac{dy}{dx} x + \frac{d^2y}{dx^2} \frac{x^2}{1.2} - \frac{d^3y}{dx^3} \frac{x^3}{1.2.3} +, \text{\&c.}$$

If in this we change A into A_1, A_2, &c., y into $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, &c., we shall, in like manner, get

$$A_1 = \frac{dy}{dx} - \frac{d^2y}{dx^2} x + \frac{d^3y}{dx^3} \frac{x^2}{1.2} - \frac{d^4y}{dx^4} \frac{x^2}{1.2.3} +, \text{\&c.},$$

$$A_2 = \frac{d^2y}{dx^2} - \frac{d^3y}{dx^3} \frac{x}{1} + \frac{d^4y}{dx^4} \frac{x^2}{1.2} -, \text{\&c.},$$

and so on, until we obtain $n - 1$ equations. If we now eliminate from the n equations which we have obtained,

$$\frac{d^ny}{dx^n}, \quad \frac{d^{n+1}y}{dx^{n+1}}, \quad \frac{d^{n+2}y}{dx^{n+2}},$$

and so on; the results will clearly be the first or $(n - 1)$th integrals of the proposed differential of the nth order, as required, which have A, A_1, A_2, &c., for their arbitrary constants.

Thus, if $n = 2$, the first integrals are

$$A = y - \frac{dy}{dx} x + f\left(x, y, \frac{dy}{dx}\right) \frac{x^2}{1.2} - f_{\prime}\left(x, y, \frac{dy}{dx}\right) \frac{x^3}{1.2.3} +, \text{\&c.},$$

and

$$A_1 = \frac{dy}{dx} - f\left(x, y, \frac{dy}{dx}\right) \frac{x}{1} + f_{\prime}\left(x, y, \frac{dy}{dx}\right) \frac{x^2}{1.2} -, \text{\&c.}$$

(See Lacroix, p. 297.)

(5.) We now propose to show how to find the integrals of differential equations of the second order between x and y, when, besides $\frac{d^2y}{dx^2}$, they contain x or y, or $\frac{dy}{dx}$, or when, besides $\frac{d^2y}{dx^2}$, they contain x and $\frac{dy}{dx}$ or y and $\frac{dy}{dx}$, by the common methods. (See Young, p. 243, &c.)

1. To integrate a differential equation of the form

$$F\left(x, \frac{d^2y}{dx^2}\right) = 0.$$

Solving the equation with reference to $\frac{d^2y}{dx^2}$, we evidently get

$$\frac{d^2y}{dx^2} = X = \text{a function of } x,$$

which, multiplied by dx and integrated, gives

$$\frac{dy}{dx} = \int X dx + C,$$

and this multiplied by dx and integrated again, gives

$$y = \int dx \int X dx + Cx + C' = \int^2 X dx^2 + Cx + C'.$$

Thus, if $X = x^n$, we have

$$\int X^2 dx^2 = \int^2 x^n dx^2 = \frac{x^{n+2}}{(n+1)(n+2)},$$

and thence $$y = \frac{x^{n+2}}{(n+1)(n+2)} + Cx + C'.$$

2. To integrate a differential equation of the form

$$F\left(y, \frac{d^2y}{dx^2}\right) = 0.$$

Here we have $\frac{d^2y}{dx^2} = Y$, which, by putting

$$\frac{dy}{dx} = p \quad \text{becomes} \quad \frac{dp}{dx} = Y,$$

whose members multiplied by

$$p = \frac{dy}{dx}, \quad \text{become} \quad p\frac{dp}{dx} = Y\frac{dy}{dx},$$

whose members multiplied by dx and then integrated, give

$$\frac{p^2}{2} = \int Y dy + C \quad \text{of the form} \quad \frac{p^2}{2} = Y' + C.$$

From this we immediately get $\frac{1}{p^2} = \frac{1}{2(Y' + C)}$, and thence

$$\frac{1}{p} = \frac{1}{\frac{dy}{dx}} = \frac{dx}{dy} = \frac{1}{\sqrt{2(Y' + C)}} \quad \text{or} \quad x = \int \frac{dy}{\sqrt{(2Y' + 2C)}},$$

which is of an integrable form.

Thus, if $Y = -\frac{y}{a^2}$, we shall have

$$Y' = -\int \frac{ydy}{a^2} = -\frac{y^2}{2a^2} + C,$$

and thence we readily get the form

$$x = \int \frac{ady}{\sqrt{(C^2 - y^2)}} = \frac{a}{C}\int \frac{dy}{\sqrt{\left\{1 - \left(\frac{y}{C}\right)^2\right\}}}$$

$$= a\int \frac{\frac{dy}{C}}{\sqrt{\left\{1 - \left(\frac{y}{C}\right)^2\right\}}} = a \sin^{-1}\frac{y}{C} + C';$$

noticing, that C^2 has been used for the first arbitrary constant.

3. To find the integral of the form $F\left(\frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0$,

or of its equivalent $F\left(p, \frac{dp}{dx}\right) = 0.$

The equation solved for $\frac{dp}{dx}$, gives $\frac{dp}{dx} = P =$ a function of p, and then $dx = \frac{dp}{P}$, an integrable form, whose integral will be expressed by $x = \int \frac{dp}{P}$; and, since $\frac{dy}{dx} = p$, we also have $dy = pdx = \frac{pdp}{P}$, an integrable form, whose integral will be expressed by the form $y = \int \frac{pdp}{P}$. Hence, eliminating p

from the equations

$$x = \int \frac{dp}{P} \quad \text{and} \quad y = \int \frac{pdp}{P},$$

we shall get the required relation between x and y. Thus, if

$$a\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2 \quad \text{or} \quad a\frac{dp}{dx} = p^2,$$

we easily get $$a\frac{dp}{p^2} = dx,$$

whose integral gives

$$C - ap^{-1} = x \quad \text{or} \quad p = \frac{a}{C - x};$$

and thence, since $dy = pdx$, we have $y = a \log C' (C - x)$ C and $a \log C'$ being the arbitrary constants.

4. To integrate the form $F\left(x, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0$,

or its equivalent, $$F\left(x, p, \frac{dp}{dx}\right) = 0.$$

This being an equation between two variables, x and p, by regarding p as being a function of x taken for the independent variable, may be integrated by the methods given in Section V., which will give the form $F(x, p, c) = 0$; in which c denotes the arbitrary constant.

If we can from this equation find p, we shall get

$$p = X = \text{a function of } x,$$

and thence $p = \frac{dy}{dx}$ gives $dy = Xdx$ or $y = \int Xdx$, whose integral gives the relation between x and y.

But if we can not find p in a function of x, or can find x more readily, then we shall get the form $x = P =$ a function of p, whose differential gives $dx = dP$; and thence from $dy = pdx$ we shall get $dy = pdP$, whose integral

$y = \int pd\mathrm{P}$ gives the relation between y and p. Hence, from the elimination of p from $x = \mathrm{P}$ and $y = \int pd\mathrm{P}$, we shall get the relation between x and y.

If F (x, p, c) can not be readily solved for p or x, we may put $\frac{dy}{dx}$ for p, and then try to integrate the result, by the methods for integrating differential equations between two variables.

Thus, $$\frac{a^2}{2x}\frac{d^2y}{dx^2} = \left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}},$$

or its equivalent $$2xdx = \frac{a^2dp}{(1 + p^2)^{\frac{3}{2}}},$$

has $$x^2 + \mathrm{C} = \frac{a^2p}{(1 + p^2)^{\frac{1}{2}}}$$

for its integral; and by subtracting the squares of the members of this from a^4, we have

$$a^4 - (x^2 + \mathrm{C})^2 = \frac{a^4}{1 + p^2}, \quad \text{or} \quad \sqrt{a^4 - (x^2 + \mathrm{C})^2} = \frac{a^2}{\sqrt{1 + p^2}};$$

consequently, dividing the members of

$$x^2 + \mathrm{C} = \frac{a^2p}{\sqrt{1 + p^2}}$$

by the corresponding members of the preceding equation,

we get $$p = \frac{x^2 + \mathrm{C}}{\sqrt{a^4 - (x^2 + \mathrm{C})^2}}, \text{ and thence}$$

$$\int dy = \int pdx = \int \frac{(x^2 + \mathrm{C})\,dx}{\sqrt{a^4 - (x^2 + \mathrm{C})^2}},$$

or $$y = \int \frac{(x^2 + \mathrm{C})\,dx}{\sqrt{a^4 - (x^2 + \mathrm{C})^2}},$$

an integrable form.

For another example, let there be taken the equation

$$1 + \left(\frac{dy}{dx}\right)^2 + x\frac{dy}{dx}\frac{d^2y}{dx^2} = a\frac{d^2y}{dx^2}\sqrt{1 + \left(\frac{dy}{dx}\right)^2},$$

or its equivalent

$$1 + p^2 + xp\frac{dp}{dx} = a\frac{dp}{dx}\sqrt{1 + p^2}.$$

Multiplying the members of this by dx and dividing the products by $1 + p^2$, we readily get

$$dx + \frac{p}{1 + p^2}xdp = \frac{a}{\sqrt{1 + p^2}}dp,$$

which is a linear equation, agreeing with the equation at p. 455, when we put x for y, and p for x, and change P and Q into

$$\frac{p}{1 + p^2} \quad \text{and} \quad \frac{a}{\sqrt{1 + p^2}}.$$

Hence, from the integral given at p. 455, we have

$$x = e^{-\int\frac{pdp}{1+p^2}}\left(a\int e^{\int\frac{pdp}{1+p^2}}\frac{dp}{\sqrt{1 + p^2}} + \text{C}\right)$$

$$= e^{-\log\sqrt{(1+p^2)}}\left(a\int e^{\log\sqrt{(1+p^2)}}\frac{dp}{\sqrt{1 + p^2}} + \text{C}\right).$$

Because

$$\frac{1}{\sqrt{1 + p^2}} = e^{-\log\sqrt{(1+p^2)}} \quad \text{and} \quad \sqrt{1 + p^2} = e^{\log\sqrt{(1+p^2)}},$$

the value of x is clearly equivalent to

$$x = \frac{ap + \text{C}}{\sqrt{1 + p^2}}.$$

By taking the differential of this, we have

$$dx = \frac{adp}{(1 + p^2)^{\frac{3}{2}}} - \frac{\text{C}pdp}{(1 + p^2)^{\frac{3}{2}}},$$

which, since $dy = pdx$, gives

$$dy = \frac{apdp}{(1+p^2)^{\frac{3}{2}}} - \frac{Cp^2\,dp}{(1+p^2)^{\frac{3}{2}}} = \frac{apdp}{(1+p^2)^{\frac{3}{2}}} - \frac{Cdp}{\sqrt{1+p^2}} + \frac{Cdp}{(1+p^2)^{\frac{3}{2}}},$$

whose integral gives

$$y = -\frac{a}{\sqrt{1+p^2}} + \frac{Cp}{\sqrt{1+p^2}} - C\log\frac{p+\sqrt{1+p^2}}{C'},$$

or $$y = \frac{Cp-a}{\sqrt{1+p^2}} - C\log\frac{p+\sqrt{1+p^2}}{C'};$$

noticing, that we here use $C\log\frac{1}{C'}$ for the arbitrary constant.

By eliminating p from x and y, the equation between x and y will be found as required.

For another example, we will find the integral of

$$2\left(a^2\frac{dy^2}{dx^2} + x^2\right)\frac{d^2y}{dx^2} = x\frac{dy}{dx},$$

or its equivalent $\quad 2(a^2p^2 + x^2)\,dp = xpdx,$

which is clearly homogeneous in p and x.

By putting $x = pz$, the equation is easily reduced to

$$2(a^2 + z^2)\,dp = z(zdp + pdz), \quad \text{or} \quad \frac{dp}{p} = \frac{zdz}{2a^2+z^2},$$

whose integral is

$$\log p = \log C\sqrt{(2a^2 + z^2)}, \quad \text{or} \quad p = C\sqrt{(2a^2 + z^2)};$$

consequently, from $x = pz$ we have

$$x = Cz\sqrt{(2a^2 + z^2)}.$$

From $x = pz$ we have

$$p = \frac{x}{z} = C\sqrt{(2a^2 + z^2)},$$

and thence, from $dy = pdx$, we have

$$dy = C\sqrt{(2a^2 + z^2)}\,dx;$$

and from $$x = Cz\sqrt{(2a^2 + z^2)},$$

which gives $$dx = C\sqrt{(2a^2 + z^2)}\,dz + \frac{Cz^2\,dz}{\sqrt{(2a^2 + z^2)}},$$

we get $$dy = C^2(2a^2\,dz + 2z^2\,dz),$$

whose integral is $$y = \frac{2}{3}C^2 z(3a^2 + z^2) + C'.$$

If we now eliminate z from the values of x and y, we shall get the sought equation between x and y, as required.

5. To integrate an equation of the form

$$F\left(y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0,$$

or its equivalent $$F\left(y, p, \frac{dp}{dx}\right) = 0.$$

Because $dx = \dfrac{dy}{p}$, we have the form reduced to

$$F\left(y, p, \frac{pdp}{dy}\right) = 0;$$

and it is clear that we may now proceed in much the same way as before. Thus, to integrate

$$\left(a + y\frac{dy}{dx}\right)\frac{d^2y}{dx^2} = \left(1 + \frac{dy^2}{dx^2}\right)\frac{dy}{dx},$$

or its equivalent

$$(a + yp)\frac{dp}{dx} = (1 + p^2)\,p.$$

By putting $\dfrac{pdp}{dy}$ for $\dfrac{dp}{dx}$, the equation is easily reduced to

$$(a + yp)\frac{dp}{dy} = 1 + p^2, \quad \text{or} \quad (a + yp)\,dp = (1 + p^2)\,dy,$$

which gives the linear equation

$$dy - \frac{p}{1 + p^2}\,ydp = \frac{a}{1 + p^2}\,dp.$$

Comparing this to the linear equation at p. 455, we have

$$P = -\frac{p}{1+p^2} \quad \text{and} \quad Q = \frac{a}{1+p^2},$$

and for dx we have dp; consequently, we shall, from the formula at p. 455, get

$$y = e^{\int \frac{pdp}{1+p^2}}\left(a\int e^{\int \frac{-pdp}{1+p^2}} \frac{1}{1+p^2}\,dp + C\right)$$

$$= e^{\log \sqrt{1+p^2}}\left(a\int e^{-\log \sqrt{1+p^2}} \frac{1}{1+p^2}\,dp + C\right);$$

or since

$$\sqrt{(1+p^2)} = e^{\log \sqrt{1+p^2}} \quad \text{and} \quad \frac{1}{\sqrt{1+p^2}} = e^{-\log \sqrt{(1+p^2)}},$$

we shall have $\quad y = ap + C\sqrt{(1+p^2)};$

and from $\quad dx = \frac{dy}{p} \quad$ we shall get

$$dx = \frac{adp}{p} + C\frac{dp}{\sqrt{(1+p^2)}},$$

whose integral is

$$x = a \log p + C \log C'(p + \sqrt{1+p^2})$$

$$= \log [p^a (C'p + C'\sqrt{1+p^2})^C].$$

Eliminating p from x and y, we shall get the sought equation between x and y.

6. When a differential equation between x and y is of the second order, and x is taken for the independent variable, then, regarding dx, dy, and d^2y each as being variables of one dimension, when the proposed equation is homogeneous in terms of its variables and their differentials, it can be reduced to a differential of the first order, which does not contain x, by putting $y = vx$ and $\frac{d^2y}{dx^2} = \frac{q}{x}$ in it. For if n

denotes the degree of homogeneity of the equation, it is plain, since $\frac{d^2y}{dx^2}$ is clearly of -1 dimensions, that it must have a factor of $n+1$ dimensions; and since vx is put for y, it is evident that wherever y is, x^n must enter as a factor; it is also evident, since $\frac{dy}{dx}$ is of no dimensions, that it must have x^n for a factor.

Because the remaining terms of the equation are of n dimensions, it is manifest that after the preceding substitutions we may divide the equation by x^n, and thus free it of x.

Thus, to find the integral of

$$xd^2y = dydx,$$

we divide its members by dx^2, and thence get

$$x\frac{d^2y}{dx^2} = \frac{dy}{dx},$$

in which we put $\frac{d^2y}{dx^2} = \frac{q}{x}$ and $\frac{dy}{dx} = p$,

and thence get $$q = p;$$

and since $\frac{d^2y}{dx^2} = \frac{q}{x}$ is the same as $\frac{dp}{dx} = \frac{q}{x}$, by putting p for q, we thence get

$$\frac{dp}{dx} = \frac{p}{x}, \quad \text{or} \quad \frac{dp}{p} = \frac{dx}{x};$$

and from $y = vx$ we have

$$dy = pdx = vdx + xdv, \quad \text{or} \quad \frac{dx}{x} = \frac{dv}{p-v},$$

and hence get

$$\frac{dp}{p} = \frac{dv}{p-v}, \quad \text{or} \quad pdp = pdv + vdp;$$

and taking the integral of this, we have

$$\frac{p^2}{2} = pv + C, \quad \text{or} \quad p^2 = 2pv + 2C.$$

Also, by taking the integrals of $\frac{dx}{x} = \frac{dp}{p}$, we have $\log x = \log pC'$, or $x = pC'$, which gives $p = \frac{x}{C'}$. Substituting this value of p in $p^2 = 2pv + 2C$, we get

$$x^2 = 2C'xv + 2CC'^2,$$

which, since $y = xv$, may clearly be represented by

$$x^2 = 2Cy + C',$$

when C and C′ represent the arbitrary constants.

Otherwise.—Since the equation

$$x\frac{d^2y}{dx^2} = \frac{dy}{dx} \quad \text{is reducible to} \quad \frac{dx}{x} = \frac{\frac{d^2y}{dx}}{\frac{dy}{dx}},$$

by taking the integral we have

$$\log x = \log C\frac{dy}{dx}, \quad \text{or} \quad x = C\frac{dy}{dx}, \quad \text{or} \quad xdx = Cdy,$$

whose integral $\frac{x^2}{2} = Cy + \text{const.}$ gives $x^2 = 2Cy + C'$, the same as by the preceding process. Again, if

$$xd^2y = ady^2 + bdx^2, \quad \text{or} \quad x\frac{d^2y}{dx^2} = a\frac{dy^2}{dx^2} + b,$$

then, putting $\frac{q}{x}$ and p for $\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$, we have $q = ap^2 + b$.

Because $\qquad \frac{dp}{dx} = \frac{q}{x}, \quad \text{or} \quad \frac{dx}{x} = \frac{dp}{q},$

by substituting the value of q, we shall have

$$\frac{dx}{x} = \frac{dp}{ap^2 + b} = \frac{dp}{b\left(1 + \frac{a}{b}p^2\right)} = \frac{dp\sqrt{\frac{a}{b}}}{\sqrt{ab}\left(1 + p^2\frac{a}{b}\right)},$$

whose integral gives

$$\log Cx = \frac{1}{\sqrt{ab}} \tan^{-1} p\sqrt{\frac{a}{b}};$$

or we have

$$Cx = e^{\frac{1}{\sqrt{ab}} \tan^{-1} p\sqrt{\frac{a}{b}}},$$

and thence

$$x = \frac{e^{\frac{1}{\sqrt{ab}} \tan^{-1} p\sqrt{\frac{a}{b}}}}{C},$$

in which e is the hyperbolic base, and C is an arbitrary constant.

Since $dy = pdx$, from $\frac{dx}{x} = \frac{dp}{ap^2 + b}$

we easily get $dy = \frac{pdp}{ap^2 + b} \times x$;

and thence, from the substitution of the value of x, we get

$$dy = \frac{e^{\frac{1}{\sqrt{ab}} \tan^{-1} p\sqrt{\frac{a}{b}}}}{C} \times \frac{pdp}{ap^2 + b};$$

and integrating this, we get y in a function of p, with a new arbitrary constant. Hence, eliminating p from the values of x and y, the equation between x and y will be determined as required.

7. Finally, it may be added that there are two classes of equations of the forms

$$F\left(\frac{dy^n}{dx^n}, \frac{d^{n-1}y}{dx^{n-1}}\right) = 0, \quad \text{and} \quad F\left(\frac{d^ny}{dx^n}, \frac{d^{n-2}y}{dx^{n-2}}\right) = 0,$$

in which n is supposed to be a positive integer greater than 2; such that they can be reduced to the forms of equations of the first and second degrees, which we have heretofore seen how to integrate.

For, by putting $\frac{d^{n-1}y}{dx^{n-1}} = u$, in the first of the preceding equations, it reduces to $F\left(\frac{du}{dx}, u\right) = 0$, which is clearly of the form of a differential equation of the first order between u and x, which gives $u = X =$ a function of x, and

$$y = \int^{n-1} X dx^{n-1}.$$

To integrate the second of the preceding equations, we put

$$\frac{d^{n-2}y}{dx^{n-2}} = u, \quad \text{and thence get} \quad \frac{d^n y}{dx^n} = \frac{d^2u}{dx^2};$$

and, by substitution, the equation becomes

$$F\left(\frac{d^2u}{dx^n}, u\right) = 0,$$

an equation of the second order, which we have seen how to integrate at p. 516. Hence, we shall have $u = X =$ a function of x; or, restoring the value of u, we have

$$\frac{d^{n-2}y}{dx^{n-2}} = X, \quad \text{which gives} \quad y = \int^{n-2} X dx^{n-2}.$$

Thus, to integrate $\frac{d^4y}{dx^4} \cdot \frac{dy^3}{dx^3} = 1$, we put $\frac{d^3y}{dx^3} = r$, and thence get $r\frac{dr}{dx} = 1$, or $rdr = dx$, for the transformed equation.

By taking the integral of the preceding equation, we have

$$\frac{r^2}{2} + C = x, \quad \text{or} \quad r = \sqrt{2(x - C)}.$$

Denoting the differential coefficient of the next inferior order to r by q, we shall clearly have

$$dq = rdx = r^2dr, \quad \text{or} \quad q = \frac{r^3}{3} + C_1,$$

and since $$dp = qdx = \left(\frac{r^3}{3} + C_1\right) rdr,$$

or $$p = \frac{r^5}{3\,.\,5} + \frac{C_1 r^2}{1\,.\,2} + C_2;$$

consequently, from

$$dy = pdx = \left(\frac{r^5}{3\,.\,5} + \frac{C_1 r^2}{1\,.\,2} + C_2\right) rdr,$$

we get $$y = \frac{r^7}{3\,.\,5\,.\,7} + \frac{C_1 r^4}{1\,.\,2\,.\,4} + \frac{C_2 r^2}{1\,.\,2} + C$$

for the complete integral, in which $r = \sqrt{2(x - C)}$.

For an example of an integral of the second order, we will take $$\frac{d^4y}{dx^4} = \frac{d^2y}{dx^2}.$$

Putting $q = \frac{d^2y}{dx^2}$, we have $\frac{d^2q}{dx^2} = \frac{d^4y}{dx^4}$,

and thence the equation is reduced to

$$\frac{d^2q}{dx^2} = q;$$

whose members, multiplied by dq, give

$$\frac{dq^2}{dx^2} = q^2 + C^2, \quad \text{or} \quad dx = \frac{dq}{\sqrt{q^2 + C^2}};$$

whose integral is $$x = \log \frac{q + \sqrt{q^2 + C^2}}{C'},$$

in which $\log \frac{1}{C'}$ is the arbitrary constant introduced by this integration. From

$$q = \frac{d^2 y}{dy^2} \quad \text{and} \quad dx = \frac{dq}{\sqrt{q^2 + C^2}}$$

we easily get $$dp = qdx = \frac{qdq}{\sqrt{q^2 + C^2}},$$

and by taking the integral

$$p = \sqrt{q^2 + C^2} + C' \quad \text{and} \quad dy = pdx = dq + C'dx,$$

whose integral is $\quad y = q + C'x + C''$;

e being the hyperbolic base,

$x = \log \frac{q + \sqrt{q^2 + C^2}}{C'}$ is equivalent to $C'e^x = q + \sqrt{q^2 + C^2}$,

which gives

$$(C'e^x - q)^2 = q^2 + C^2, \quad \text{or} \quad C'^2 e^{2x} - 2C'e^x q = C^2,$$

and thence $\quad q = \frac{C'e^x}{2} - \frac{C^2}{2C'} e^{-x}.$

From the substitution of q in $y = q + C'x + C''$, we have

$$y = \frac{C'e^x}{2} - \frac{C^2}{2C'} e^{-x} + C'x + C'',$$

which may more readily be represented by

$$y = Ce^x + C_1 e^{-x} + C_2 x + C_3,$$

C, C_1, C_2, and C_3 being the arbitrary constants.

(6.) We will now show how to find the integral of a *linear equation* of the nth order, represented by

$$\frac{d^n y}{dx^n} + A \frac{d^{n-1} y}{dx^{n-1}} + B \frac{d^{n-2} y}{dx^{n-2}} + \ldots\ldots + M \frac{dy}{dx} + Ny + X = 0,$$

in which the coefficients A, B, C, &c., may either be constants, or they may contain the independent variable x without y. We shall determine the general form of the sought integral by integrating the more simple equation

$$\frac{d^n y}{dx^n} + A \frac{d^{n-1} y}{dx^{n-1}} + B \frac{d^{n-2} y}{dx^{n-2}} + \ldots\ldots + M \frac{dy}{dx} + Ny = 0,$$

which is independent of X, and the coefficients are supposed to be constants.

Supposing e to be the base of hyperbolic logarithms, C and m constants to be determined; if we represent y by Ce^{mx} by the rule at p. 56, we shall have

$$dy = \mathrm{C}de^{mx} = m\mathrm{C}e^{mx}dx, \quad \text{or} \quad \frac{dy}{dx} = m\mathrm{C}e^{mx},$$

and in like manner

$$\frac{d^2y}{dx^2} = m^2\,\mathrm{C}e^{mx}, \quad \frac{d^3y}{dx^3} = m^3\mathrm{C}e^{mx},$$

and so on, to any required extent. Hence, by substituting the values of $\frac{d^ny}{dx^n}, \frac{d^{n-1}y}{dx^{n-1}}, \frac{d^{n-2}y}{dx^{n-2}}$, &c., in the preceding equation, and rejecting the factors C and e^{mx}, which will be common to all the terms of the resulting equation, we shall get the algebraic equation of the nth degree,

$$m^n + \mathrm{A}m^{n-1} + \mathrm{B}m^{n-2} + \ldots\ldots + \mathrm{M}m + \mathrm{N} = 0,$$

which, from the well-known theory of equations, must have n roots. Solving the equation, we shall have the n roots, which may be denoted by m_1, m_2, m_3, and so on, to m_n inclusive; and since each of these roots satisfies the equation, if $\mathrm{C}_1, \mathrm{C}_2$, C_3, and so on, to C_n inclusive, denote the const., then, if the roots are unequal, we shall clearly have

$$y = \mathrm{C}_1e^{m_1x} + \mathrm{C}_2e^{m_2x} + \mathrm{C}_3e^{m_3x} + \ \ldots\ + \mathrm{C}_{n-1}e^{m_{(n-1)}x} + \mathrm{C}_ne^{m_nx},$$

for the complete integral of the proposed equation; as is manifest from the circumstance that each of its terms satisfies it, and of course all its terms, conjunctly, will satisfy it; and, since y contains n constants, or a number equal to the order of the proposed differential equation, it evidently results that the preceding equation between x and y must be the complete integral required. It may be added, that, so long as the roots of the equation are unequal, the integral will be of the preceding form, whether the roots are all real, or some (an even number) of them are imaginary. If two of the roots of the equation, as m_1 and m_2, are imaginary, they may be expressed by the forms

$$a + b\sqrt{-1} \quad \text{and} \quad a - b\sqrt{-1},$$

since (see p. 440 of my Algebra) the imaginary roots of equations are known to occur in pairs of the preceding forms; consequently, for the terms $C_1e^{m_1x} + C_2e^{m_2x}$ of y, we may write

$$C_1e^{(a+b\sqrt{-1})x} + C_2e^{(a-b\sqrt{-1})x} = e^{ax}(C_1e^{b\sqrt{-1}x} + C_2e^{-b\sqrt{-1}x}),$$

and since (see p. 58),

$$e^{bx\sqrt{-1}} = \cos bx + \sqrt{-1}\sin bx$$

and $$e^{-bx\sqrt{-1}} = \cos bx - \sqrt{-1}\sin bx,$$

this is easily reduced to

$$e^{ax}[(C_1 + C_2)\cos bx + (C_1 - C_2)\sqrt{-1}\sin bx].$$

Since we may clearly assume such values for the constants C_1 and C_2, that we shall have

$$C_1 + C_2 = p\sin q \quad \text{and} \quad (C_1 - C_2)\sqrt{-1} = p\cos q,$$

we shall thence change the preceding expression into

$$pe^{ax}\sin(bx + q).$$

Hence it results that we have reduced the terms

$$C_1e^{m_1x} + C_2e^{m_2x} \text{ of } y \quad \text{to the form} \quad pe^{ax}\sin(bx + q),$$

and it is clear that every other corresponding pair of imaginary roots will admit of like reductions.

It may be noticed, that if our equation has two or more equal roots, our method of finding the equation between x and y will fail for the equal roots, and will be applicable only to the unequal roots. Thus, if in the equation between

x and y we suppose the root m_1 equals the root m_2, the equation will become

$$y = (C_1 + C_2)\, e^{m_1 x} + C_3 e^{m_3 x} +, \&c.,$$

which clearly shows that the two constants C_1 and C_2 are actually only equivalent to a single constant represented by $C_1 + C_2$, so that the equation will be defective in not having a sufficient number of constants, which ought to equal n, the order of the proposed differential equation.

The defect may easily be remedied by writing the terms containing the equal roots in the form

$$C_1 e^{m_1 x} + C_2 x e^{m_1 x};$$

if three roots, as m_1, m_2, m_3, are equal, they must be written in the form

$$C_1 e^{m_1 x} + C_2 x e^{mx} + C_3 x^2 e^{m_1 x},$$

and so on; observing, that the index of x in the last of the terms of this kind is less by a unit than the number of equal roots. To be satisfied as to the correctness of what has been said, it will be sufficient to apply it to the integral of the

equation $$\frac{d^3y}{dx^3} + A\frac{d^2y}{dx^2} + B\frac{dy}{dx} + Cy = 0,$$

which is clearly a particular case of the general differential equation, obtained from it by putting 3 for n in it; consequently, the algebraic equation at p. 530 will here become

$$m^3 + Am^2 + Bm + C = 0,$$

which, we shall suppose, has a pair of equal roots represented by $m_1 = m_2$; then, shall

$$y = C_1 e^{m_1 x}, \quad y = C_2 x e^{m_1 x}, \quad \text{or} \quad y = C_1 e^{m_1 x} + C_2 x e^{m_1 x},$$

each satisfy the preceding equation of the third order.

To show what is here said, it will evidently be sufficient to show that the differential equation is satisfied by putting $y = C_2 x e^{m_1 x}$, which gives

$$\frac{dy}{dx} = C_2 m_1 x e^{m_1 x} + C_2 e^{m_1 x}, \quad \frac{d^2y}{dx^2} = C_2 m_1^2 x e^{m_1 x} + 2C_2 m_1 e^{m_1 x},$$

and
$$\frac{d^3y}{dx^3} C_2 m_1^3 x e^{m\ x} + 3C_2 m_1^2 e^{m_1 x}.$$

Hence, from the substitution of these values in

$$\frac{d^3y}{dx^3} + A\frac{d^2y}{dx^2} + B\frac{dy}{dx} + Cy = 0,$$

we get

$$C_2 x e^{m_1 x}(m_1^3 + Am_1^2 + Bm_1 + C) +$$
$$C_2 e^{m_1 x}(3m_1^2 + 2Am_1 + B) = 0;$$

consequently, because m_1 is one of the equal roots of the algebraic equation at p. 532, we have

$$m_1^3 + Am_1^2 + Bm_1 + C = 0,$$

and because
$$3m_1^2 + 2Am_1 + B$$

is the first derived or limiting equation of this, one of its roots must also equal m^1, and thence we also have

$$3m_1^2 + 2Am_1 + B = 0, \quad \text{and thence} \quad y = C_2 x e^{m_1 x}$$

satisfies the equation as it ought to do. (See any of the treatises on Algebra, on the equal roots of equations.)

It is hence easy to perceive that

$$y = C_1 e^{m_1 x} + C_2 x e^{m_1 x} + C_3 e^{m_3 x},$$

must satisfy the equation

$$\frac{d^3y}{dx^3} + A\frac{d^2y}{dx^2} + B\frac{dy}{dx} + Cy = 0,$$

since each of its terms satisfies it; and because it contains three arbitrary constants it must be the complete integral of the equation.

Because a reasoning similar to the above is applicable to each term of y that results from any number of equal roots in the equation at p. 530, it follows that the terms of y resulting from any number of equal roots will be represented according to the directions given above.

If the equation at p. 530 has four imaginary roots, such, that two corresponding roots of each form, as $a + b\sqrt{-1}$ and $a - b\sqrt{-1}$, are equal; then, according to what has been shown at p. 531, and to what has been shown above, they may be expressed by the forms

$$e^{ax}\left[(C_1 + C_2x)\cos bx + (C_3 + C_4x)\sin bx\right];$$

and it is clear that we may proceed in much the same way, when the equation contains any number of pairs of equal imaginary roots.

Hence, having found

$$y = C_1e^{m_1x} + C_2e^{m_2x} + C_3e^{m_3x} + \ldots\ldots + C^ne^{m_nx}$$
$$= C_1y_1 + C_2y_2 + C_3y_3 + \ldots\ldots + C^ny^n,$$

for the complete integral of

$$\frac{d^ny}{dx^n} + A\frac{d^{n-1}y}{dx^{n-1}} + B\frac{d^{n-2}y}{dx^{n-2}} + \ldots\ldots M\frac{dy}{dx} + Ny = 0,$$

whether A, B, &c., are functions of x or not, y_1, y_2, &c., being called particular values of y, since each of them is supposed to satisfy the above equations; then, we may assume

$$y = C_1y_1 + C_2y_2 + \ldots\ldots + C^ny$$

for the complete integral of

$$\frac{d^ny}{dx^n} + A\frac{d^{n-1}y}{dx^{n-1}} + B\frac{d^{n-2}y}{dx^{n-2}} + \ldots\ldots M\frac{dy}{dx} + Ny + X = 0,$$

in which X is supposed to be a function of x; by supposing the arbitrary constants C_1, C_2, C_3, and so on, to vary, by subjecting them to the following conditions, which we will illustrate by the case of $n = 3$.

Thus, the equation to be integrated is

$$\frac{d^3y}{dx^3} + A\frac{d^2y}{dx^2} + M\frac{dy}{dx} + Ny + X = 0,$$

whose integral we suppose to be represented by

$$y = C_1y_1 + C_2y_2 + C_3y_3,$$

supposed to be subjected to the following conditions:—

1st. We have

$$\frac{dy}{dx} = C_1\frac{dy^1}{dx} + C_2\frac{dy^2}{dx} + C^3\frac{dy^3}{dx},$$

by assuming $y_1dC_1 + y_2dC_2 + y_3dC_3 = 0$.

2d. We have

$$\frac{d^2y}{dx^2} = C_1\frac{d^2y_1}{dx^3} + C_2\frac{d^2y_2}{dx^2} + C_3\frac{d^2y_3}{dx^2},$$

by assuming the equation

$$dC_1dy_1 + dC_2dy_2 + dC_3dy_3 = 0.$$

3d. We have

$$\frac{d^3y}{dx^3} = C_1\frac{d^3y_1}{dx^3} + C_2\frac{d^3y_2}{dx^3} + C_3\frac{d^3y_2}{dx^3} - X,$$

by assuming

$$\frac{dC_1d^2y_1}{dx^3} + \frac{dC_2d^2y_2}{dx^3} + \frac{dC_3d^2y_3}{dx^3} + X = 0.$$

Hence, if we determine the constants from these three conditions, we shall have

$$y = C_1y_1 + C_2y_2 + C_3y_3$$

for the complete integral, since it will (generally) contain three constants, as it ought to do. It is manifest that we may proceed in the same way to find the integral, when the proposed differential equation consists of any number of terms, or is of any order.

REMARKS.—1. For the preceding beautiful method of finding the integral of

$$\frac{d^n y}{dx^n} + A\frac{d^{n-1}y}{dx^{n-1}} + B\frac{d^{n-2}y}{dx^{n-2}} + \ldots\ldots + M\frac{dy}{dx} + Ny + X = 0$$

from that of the same equation, when X is omitted, by means of the *variations of its arbitrary constants*, we must refer the reader to p. 323, vol. i., of the "Mécanique Analytique" of Lagrange, the inventor of the process; reference may also be made to p. 326, vol. ii., of Lacroix; to Young, and to most of the late writers on physical astronomy.

2. On account of its importance in determining the lunar motions, we propose to give a different and very simple method of finding the integral of

$$\frac{d^2u}{dv^2} + m^2u + P = 0,$$

which is an equation of the second order, in which P may involve v or be independent of it, according to the nature of the case.

By putting

$\sin mv = s$ and $\cos mv = s'$, we have $s^2 + s'^2 = 1$,

which gives $sds + s'ds' = 0$;

and by taking the second differentials of the equations

$$\sin mv = s \quad \text{and} \quad \cos mv = s',$$

we also have the equations

$$\frac{d^2s}{dv^3} + m^2s = 0 \quad \text{and} \quad \frac{d^2s'}{dv^2} + m^2s' = 0.$$

Multiplying the proposed equation by ds, and the first of these by du, and adding the products, we have

$$\frac{d(dsdu)}{dv^2} + m^2 d(su) + Pds = 0,$$

whose integral gives

$$\frac{dsdu}{dv^2} + m^2 su + \int Pds = mA = \text{const.};$$

and in like manner, from the proposed and the second of the preceding equations, we have

$$\frac{ds'du}{dv^2} + m^2 s'u + \int Pds' = mB = \text{const.}$$

Multiplying the first of these by s, and the second by s', and adding the products, since

$$sds + s'ds' = 0 \quad \text{and} \quad s^2 + s'^2 = 1,$$

we get $m^2 u + s\int Pds + s'\int Pds' = m(As + Bs')$,

or $$mu = A \sin mv +$$

$$B \cos mv - \sin mv \int P \cos mv dv + \cos mv \int P \sin mv dv,$$

for the sought integral. If

$$P = -\frac{M + m}{C^2}, \quad u = \frac{1}{r},$$

such that M and m represent the masses of the sun and a planet revolving round each other at the distance r, C^2 representing the square of twice the area they describe around each other in the unit of time, and v the angle r makes with a fixed line, then, if $m = 1$, our integral becomes

$$u = \frac{1}{r} = A \sin v + B \cos v + \frac{M + m}{C^2}.$$

If in this we put

$C^2 = (M + m)p$, $Ap = e \cos w$, and $Bp = e \sin w$,

we shall get $$r = \frac{p}{1 + e \cos (v - w)}$$

for the equation of the curve described by m in its revolution around M, regarded as being at rest, which is clearly an ellipse when e is less than unity. (See Whewell's "Dynamics," p. 27.)

(7.) If a differential equation between x and y involves only the simple powers of y and dy in separate terms, and has other terms that are independent of y and dy, which do not involve fractional or negative powers of x, then the proposed equation may be greatly simplified by differentiating it successively on the supposition that y is a function of x regarded as the independent variable.

For, since (according to what is shown at pp. 11 to 13) dx is constant, the terms that do not contain y and dy will disappear from the equation in consequence of its successive differentiations.

It is hence manifest, that if we integrate the first of the differential equations that is freed from the preceding terms, we shall (often) readily find the integral of the proposed equation. Thus, taking

$$ady = ydx + Cx^2dx, \quad \text{or its equivalent} \quad \frac{ady}{dx} - y = Cx^2$$

(from p. 217 of Simpson's "Fluxions"), by taking its successive differential coefficients, regarding x as being the independent variable, or dx as constant, we have

$$a\frac{d^2y}{dx^2} - \frac{dy}{dx} = 2Cx, \quad a\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} = 2C, \quad \text{and} \quad a\frac{d^4y}{dx^4} - \frac{d^3y}{dx^3} = 0.$$

thence to find the integral of the proposed equation, we must find y, such that it shall satisfy

$$a\frac{d^4y}{dx^4} - \frac{d^3y}{dx^3} = 0.$$

If e represents the base of hyperbolic logarithms, and D an arbitrary constant, then $y = De^{mx}$, in which m is constant, reduces the preceding equation to

$$aDm^4e^{mx} - Dm^3e^{mx} = 0,$$

which gives $m = \frac{1}{a}$; and thence $y = De^{\frac{x}{a}}$, is the first integral of the preceding equation, and of course a part of the integral of the proposed equation.

Since $$a\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} = 2C$$

is clearly the first direct integral of

$$a\frac{d^4y}{dx^4} - \frac{d^3y}{dx^3} = 0,$$

having 2C for the constant, it is manifest that $y = De^{\frac{x}{a}}$, having D for its constant, must represent what is called an indirect integral of the same equation.

To find the remaining terms of y, or the proposed integral, since it is clearly to be regarded as the integration of an equation analogous to

$$a\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} = 2C,$$

which being of the third order of differentials, its complete integral must contain three arbitrary constants; consequently, we may represent the sought terms by

$$y = Ax^2 + Bx + C,$$

in which A, B, C, are the constants, and the integral is evidently of the proper form, since it must be supposed to have vanished from the equation

$$a\frac{d^4y}{dx^4} - \frac{d^3y}{dx^3} = 0,$$

in consequence of the successive differentiations of the proposed equation.

To find A, B, C, we substitute the values of y and $\frac{dy}{dx}$ for them in the equation

$$a\frac{dy}{dx} - y = cx^2,$$

and thence get

$$(A + c)\,x^2 + (B - 2aA)\,x + C - aB = 0,$$

which must clearly be an identical equation; consequently, we must have

$$A + c = 0, \quad B - 2aA = 0, \quad C - aB = 0;$$

which give

$$A = -c, \quad B = 2aA = -2ac, \quad C = aB = -2a^2c.$$

Hence, from the substitution of these values, we have

$$y = Ax^2 + Bx + C = -c\,(x^2 + 2ax + 2a^2);$$

consequently, adding this to the preceding value of y, we have

$$y = De^{\frac{x}{a}} - c\,(x^2 + 2ax + 2a^2)$$

for the complete integral of the proposed equation, which is the same as found by Simpson.

By a like reasoning, the integral of the differential equation

$$ady = ydx + cx^n dx,$$

will be found to be expressed by

$$y = De^{\frac{x}{a}} - c$$

$$[x^n + nax^{n-1} + n\,(n-1)\,a^2x^{n-2} + n\,(n-1)\,(n-2)\,a^3x^{n-3} + \text{\&c.}],$$

which agrees with Simpson's integral; noticing, that the integral will consist of an unlimited number of terms, when n is not a positive whole number.

(8.) The integral of a differential equation of the second order between x and y is often readily effected by interchanging the dependent and independent variables, or, which is the same (see p. 36), since $\frac{d^2y}{dx}$ and $\frac{d^2x}{dy}$ are equivalent to

$$d\left(\frac{dy}{dx}\right) = -\frac{dyd^2x}{dx^2} \text{ and } d\left(\frac{dx}{dy}\right) = -\frac{dxd^2y}{dy^2},$$

by writing

$$-\frac{dyd^2x}{dx} \text{ for } d^2y, \text{ or } -\frac{dxd^2y}{dy} \text{ for } d^2x.$$

Thus, by taking

$$dxdy - xd^2y - ad^2y - \frac{xdy^2}{b} = 0$$

(from p. 183 of Vince's "Fluxions"), and writing $-\frac{dyd^2x}{dx}$ for d^2y in it, we get

$$dxdy + \frac{xdyd^2x}{dx} + \frac{adyd^2x}{dx} - \frac{xdy^2}{b} = 0,$$

or its equivalent

$$dx^2 + xd^2x + ad^2x - \frac{xdxdy}{b} = 0.$$

Because $\quad dx^2 + xd^2x = d(xdx),$

and that dy is constant or invariable, by taking the integral of the preceding equation, after dividing by dy, we get

$$\frac{xdx}{dy} + \frac{adx}{dy} - \frac{x^2}{2b} = \text{C},$$

an arbitrary constant. Since this equation gives

$$dy = \frac{2bxdx}{2bc + x^2} + \frac{2abdx}{2bc + x^2},$$

by taking the integrals of this, and using C′ for the arbitrary constant, we have

$$y + C' = \log (2bC + x^2)^b + a\sqrt{\frac{2b}{C}} \times \text{arc}\left(rd = 1 \text{ and } \tan \frac{x}{\sqrt{2bC}}\right)$$

which agrees with Vince's integral after the constant C′ is added to his value of y.

For another example, we will take the equation

$$xd^2x - dydx = 0,$$

which is such that the method of integration does not readily present itself to the mind.

By substituting $-\frac{dxd^2y}{dy}$ for d^2x in this equation, it becomes

$$-\frac{dxd^2y}{dy}x - dydx = 0 \quad \text{or} \quad dx\frac{d^2y}{dy^2} + \frac{dx}{x} = 0,$$

in which dx is constant or invariable. Since this equation is the same as

$$-dxd\left(\frac{1}{dy}\right) + d(\log x) = 0,$$

by integrating and using C for the arbitrary constant, we have

$$-\frac{dx}{dy} + \log x = C \quad \text{or} \quad dy = \frac{dx}{\log x - C},$$

in which the variables are separated, and

$$y = \int \frac{dx}{\log x - C}$$

indicates the integral. Putting

$$\log x - C = z, \quad \text{we get} \quad \frac{dx}{x} = dz \quad \text{or} \quad dx = xdz,$$

which, since $\log x = C + z$ gives $x = e^{C+z} = e^C e^z$, and reduces the preceding integral to

$$y = e^{C}\int\frac{e^{z}dz}{z},$$

which (see p. 51), gives

$$y = e^{C}\int\left(\frac{dz}{z} + \frac{dz}{1} + \frac{zdz}{1.2} + \frac{z^2dz}{1.2.3} + \&c.\right)$$

$$= e^{C}\left(\log z + \frac{z}{1} + \frac{1}{2}\cdot\frac{z^2}{1.2} + \frac{1}{3}\cdot\frac{z^3}{1.2.3} + \frac{1}{4}\cdot\frac{z^4}{1.2.3.4} + \&c.\right) + C'',$$

in which C′ is the arbitrary constant. If $y = 0$ when $z = 1$, and

$$1 + \frac{1}{2}\cdot\frac{1}{1.2} + \frac{1}{3}\cdot\frac{1}{1.2.3} + \frac{1}{4}\cdot\frac{1}{1.2.3.4} + \&c. = 1.3179021513 + = a,$$

we have $C' = -ae^{C}$, and thence

$$y = e^{C}\left(\log z + \frac{z}{1} + \frac{1}{2}\cdot\frac{z^2}{1.2} + \frac{1}{3}\cdot\frac{z^3}{1.2.3} + \&c. - a\right),$$

which will clearly give all the values of y that correspond to those values of z or $\log x - C$ that do not differ greatly from unity, or that are positive and not very small. (See Lacroix, p. 512, vol. iii.)

(9.) Sometimes a differential equation can be integrated more easily by eliminating an arbitrary constant from it, by means of its differential equation, particularly when it contains two variables, as x and y, and higher powers of $\frac{dy}{dx}$ or $\frac{dx}{dy}$ than the first power.

Thus, by taking the differential of

$$Axy\frac{dy^2}{dx^2} + (x^2 - Ay^2 - B)\frac{dy}{dx} - xy = 0,$$

we have

$$\frac{d^2y}{dx^2}\left(2Axy\frac{dy}{dx} + x^2 - Ay^2 - B\right) + \left(A\frac{dy^2}{dx^2} + 1\right)\left(x\frac{dy}{dx} - y\right) = 0,$$

in which x is taken for the independent variable. From the proposed equation we get

$$x^2 - Ay^2 - B = xy\,\frac{\left(1 - A\dfrac{dy^2}{dx^2}\right)}{\dfrac{dy}{dx}},$$

which being substituted in the preceding differential, and rejecting the useless factor $A\dfrac{dy^2}{dx^2} + 1$, gives the differential equation

$$xyd^2y + dy\,(xdy - ydx) = 0, \quad \text{or} \quad \frac{x}{y}\,d^2y - dyd\,\frac{x}{y} = 0.$$

The first integral of this clearly is

$$\frac{dy}{dx} = C\,\frac{x}{y},$$

and the integral of this is

$$y^2 = Cx^2 + C'.$$

From the substitution of the values of $\dfrac{dy}{dx}$ and y^2 in the proposed equation, after a slight reduction, we get

$$ACC' + C' = -BC, \quad \text{and thence} \quad C' = -\frac{BC}{AC + 1};$$

consequently, from the substitution of this, we shall get

$$y^2 = Cx^2 - \frac{BC}{AC + 1}$$

for the integral of the proposed equation.

Remark.—For the substance of what has here been done, we shall refer to p. 123, &c., of Monge's "Application de l'Analyse à la Géométrie," and to Lacroix, p. 370, vol. ii.

(10.) The integral of a differential, or differential equation, between x and y, may sometimes be found by assuming an expression for the integral, or for the relation between x and

y, in undetermined coefficients and exponents of x and y if required, such, that by putting the differential of the assumed integral equal to that proposed, they may be made identical, so as to determine the indices and coefficients.

Thus, to find the integral of the differential

$$\frac{adx + bxdx}{cx + x^2}$$

(given by Vince at p. 184 of his "Fluxions"), it is evident that it may be represented by the form

$$A \log (cx^r + x^{r+1}),$$

of like form to the integral assumed by Vince, in which A and r are to be found.

By taking the differential of the assumed integral, we have

$$A \times \frac{rcx^{r-1}dx + (r + 1)\,x^r dx}{cx^r + x^{r+1}},$$

which must be made identical to the proposed differential which serves to find the unknown A and r. If we put $r = s + 1$, and multiply the numerator and denominator of the proposed differential by x^s, its denominator becomes identical to that of the assumed differential; consequently, we must have the identical equation

$$(ax^s + bx^{s+1})\,dx = [Arcx^s + A\,(r + 1)\,x^{s+1}]\,dx,$$

or, rejecting the useless factor $x^s dx$, we shall have the identical equation

$$a + bx = Arc + A\,(r + 1)\,x,$$

which, by the method of undetermined coefficients, gives $a = Arc$ and $b = A\,(r + 1)$, and thence $\dfrac{b}{a} = \dfrac{r + 1}{rc}$, which gives

$$r = \frac{a}{bc - a} \quad \text{and} \quad A = \frac{a}{rc} = \frac{bc - a}{c}.$$

From the substitution of the values of r and Λ in the assumed integral, and using $\frac{bc-a}{c}\log\frac{1}{C}$ to stand for the arbitrary constant, we get

$$\int\frac{adx+bxdx}{cx+x^2}=\frac{bc-a}{c}\log\frac{cx^{\frac{a}{bc-a}}+x^{\frac{bc}{bc-a}}}{C}$$

for the correct integral. Putting

$$C=cx'^{\frac{a}{bc-a}}+x'^{\frac{bc}{bc-a}},$$

in which x' represents some particular value of x, then

$$\int\frac{adx+bxdx}{cx+x^2}=\log\left\{\frac{cx^{\frac{a}{bc-a}}+x^{\frac{bc}{bc-a}}}{cx'^{\frac{a}{bc-a}}+x'^{\frac{bc}{bc-a}}}\right\}^{\frac{bc-a}{c}},$$

which equals naught when $x=x'$, and agrees with Vince's integral when it is properly corrected.

(11.) Sometimes the integral of a differential equation of the higher orders, between x and y, may be simplified in form by taking a function of the independent variable for a new independent variable. Thus, if we take

$$\frac{d^2y}{dx^2}+\frac{A}{x}\frac{dy}{dx}-B^2x^ny=0,$$

and put $$u=x^{n+2}=x^m,$$

we may evidently take u for the independent variable.

Because y is supposed to be a function of x, which equals a function of u, by taking the differentials of these equal functions, we have

$$\frac{dy}{dx}dx=\frac{dy}{du}du,\quad\text{or}\quad\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx};$$

consequently, since $u = x^m$ gives

$$\frac{du}{dx} = mx^{m-1},$$

we shall get $$\frac{dy}{dx} = m\frac{dy}{du}x^{m-1}.$$

Again, since u is to be taken for the independent variable, we must, for $\frac{d^2y}{dx^2}$ in the proposed equation (see p. 36), write its equivalent $\frac{d\left(\frac{dy}{dx}\right)}{dx}$; consequently, since

$$\frac{dy}{dx} = m\frac{dy}{du}x^{m-1},$$

and that $\frac{dy}{du}$ is a function of x.

$$\frac{d\left(\frac{dy}{dx}\right)}{dx} = m(m-1)\frac{dy}{du}x^{m-2} + \frac{d^2y}{du^2}\cdot\left(\frac{du}{dx}\right)^2$$

$$= m(m-1)\frac{dy}{du}x^{m-2} + m^2\frac{d^2y}{du^2}x^{2m-2}.$$

Hence, from the substitutions of these values of

$$\frac{dy}{dx}, \quad \frac{d\left(\frac{dy}{dx}\right)}{dx},$$

in the proposed equation, after an obvious reduction, it will become $$\frac{d^2y}{du^2} + \left(1 + \frac{A-1}{m}\right)\frac{dy}{udu} - \frac{B^2y}{m^2u} = 0;$$

which, by putting $1 + \frac{A-1}{m} = c$ and $\frac{B^2}{m^2} = h$, becomes

$$\frac{d^2y}{du^2} + c\frac{dy}{udu} - h\frac{y}{u} = 0;$$

which is the sought transformation, and it is clearly of a much simpler form than the proposed equation.

For another example we will take the equation

$$\frac{d^2y}{dx^2} + \text{A}\frac{dy}{dx} - \text{B}^2 e^{nx} y = 0,$$

and shall take $u = e^{nx}$ for the independent variable.

From $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$, we have $\dfrac{dy}{dx} = n\dfrac{dy}{du}e^{nx}$,

and thence $$\frac{d\left(\frac{dy}{dx}\right)}{dx} = n^2\frac{d^2y}{du^2}e^{2nx} + n^2\frac{dy}{du}e^{nx};$$

consequently, from the substitutions of these values, the proposed equation becomes

$$\frac{d^2y}{du^2} + \left(1 + \frac{\text{A}}{n}\right)\frac{dy}{udu} - \frac{\text{B}^2}{n^2}\frac{y}{u} = 0,$$

which, by putting

$$1 + \frac{\text{A}}{n} = c \quad \text{and} \quad \frac{\text{B}^2}{n^2} = h,$$

becomes $$\frac{d^2y}{du^2} + c\frac{dy}{udu} - h\frac{y}{u} = 0,$$

which is the same as the transformation in the preceding example; consequently, the solution of one of the given equations must be given by that of the other.

REMARK.—The preceding equations are due to Professor Peirce; they appear to have been first published at p. 399 of Professor Gill's "Mathematical Miscellany."

(12.) When a differential equation between x and y is such, that its integral, in finite terms, can not easily be found, then we express the dependent variable in a series of terms of the independent variable, having undetermined exponents and coefficients; and then, substituting the assumed series for the dependent variable in the differential equation, we determine the exponents and coefficients, so that the indices of the independent variable shall increase from left to right for an ascending series, and shall decrease from left to right for a descending series.

Thus, to integrate

$$\frac{d^2y}{du^2} + c\,\frac{dy}{udu} - h\,\frac{y}{u} = 0,$$

the transformation found in (11), it is easy to perceive that we may assume

$$y = \mathrm{A}u^a + \mathrm{B}u^{a+1} + \mathrm{C}u^{a+2} +, \text{\&c.},$$

which, being put for y and its differentials for those of d^2y and dy in the equation, gives

$$\mathrm{A}\,[a\,(a+c-1)]\,u^{a-2} + [\mathrm{B}\,(a+1)\,(a+c) - \mathrm{A}h]\,u^{a-1} +$$
$$[\mathrm{C}\,(a+2)\,(a+c+1) - \mathrm{B}h]\,u^a +$$
$$[\mathrm{D}\,(a+3)\,(a+c+2) - \mathrm{C}h]\,u^{a+1} + \text{\&c.} = 0,$$

which must clearly be an identical equation, or be satisfied independently of u. The first term of the equation is evidently reduced to naught by putting

$$a = 0, \quad \text{or} \quad a + c - 1 = 0,$$

which gives $a = 1 - c$, and A is arbitrary. It is also manifest that the remaining terms of the equation will be reduced to naught by the equations

$$\mathrm{B} = \frac{\mathrm{A}h}{(a+1)\,(a+c)}, \quad \mathrm{C} = \frac{\mathrm{B}h}{(a+2)\,(a+c+1)},$$

$$\mathrm{D} = \frac{\mathrm{C}h}{(a+3)\,(a+c+2)},$$

and so on. If in these expressions we put $a = 0$, they will give $B = \frac{Ah}{1 . c}$, $C = \frac{Bh}{1 . 2\,(c + 1)} = \frac{Ah^2}{1 . 2c\,(c + 1)}$,

$$D = \frac{Ah^3}{1 . 2 . 3c\,(c + 1)\,(c + 2)},$$

and so on; and in a similar way, by putting $a = 1 - c$, and using A′ for the corresponding value of A, the same expressions will give

$$B' = \frac{A'h}{1 . (2 - c)}, \quad C' = \frac{A'h^2}{1 . 2 . (2 - c)\,(3 - c)},$$

$$D' = \frac{A'h^3}{1 . 2 . 3 . (2 - c)\,(3 - c)\,(4 - c)},$$

and so on; which, by putting $2 - c = C'$, become

$$B' = \frac{A'h}{1 . c'}, \quad C' = \frac{A'h^2}{1 . 2 . c'\,(c' + 1)},$$

$$D' = \frac{A'h^3}{1 . 2 . 3 . c'\,(c' + 1)\,(c' + 2)},$$

&c., which represent the values of B, C, &c., that correspond to A′.

Hence, according to principles heretofore given, from the substitution of the preceding particular values in the assumed value of y, we shall get the complete value of y expressed by

$$y = A\left\{1 + \frac{hu}{1.c} + \frac{h^2u^2}{1.2.c\,(c+1)} + \frac{h^3u^3}{1.2.3.c\,(c+1)\,(c+2)} + \&c.\right\} +$$

$$A'u^{c'-1}\left\{1 + \frac{hu}{1.c'} + \frac{h^2u^2}{1.2.c'(c'+1)} + \frac{h^3u^3}{1.2.3.c'(c'+1)\,(c'+2)}\ \&c.\right\},$$

in which A and A′ represent the two arbitrary constants, which the complete integration of the proposed differential equation requires.

If in this integral we put the values of u, c, and h, that correspond to the equations

$$\frac{d^2y}{dx^2} + \frac{A}{x}\frac{dy}{dx} - B^2x^ny = 0, \quad \text{and} \quad \frac{d^2y}{dx^2} + A\frac{dy}{dx} - B^2e^{nx}y = 0,$$

separately, it is clear, from what is shown in (11), that we shall get their integrals. Hence, if in the first of these integrals we put $A = 0$, or $c = 1 - \frac{1}{m}$, it will be the integral of the equation

$$\frac{d^2y}{dx^2} - B^2x^ny = 0;$$

and by putting $A = 0$, or $c = 1$, the second of the preceding integrals will be the integral of

$$\frac{d^2y}{dx^2} - B^2e^{nx}y = 0.$$

It may be added, that if we put $y = e^{\int tdx}$ (e being the hyperbolic base), we shall have

$$\frac{dy}{dx} = te^{\int tdx}, \quad \text{and thence} \quad \frac{d^2y}{dx^2} = (dt + t^2dx)\, e^{\int tdx};$$

consequently, the equation

$$\frac{d^2y}{dx^2} - B^2x^ny = 0$$

is immediately reducible to

$$dt + t^2dx - B^2x^n = 0,$$

which agrees with the equation of Riccati. (See Lacroix, pp. 256 and 417, vol. ii.; and Young, p. 202.)

From $y = e^{\int tdx}$, or its equivalent $\log y = \int tdx$, we get $t = \frac{dy}{ydx}$; consequently, if with

$$y = \mathrm{A}\left(1 + \frac{hu}{1 \, . \, c} + \&c.\right) + \mathrm{A}' u^{c'-1}\left(1 + \frac{hu}{1 \, . \, c'} + \&c.\right),$$

the integral of $\qquad \dfrac{d^2y}{dx^2} - \mathrm{B}^2 x^n y = 0,$

after x^{m+2} has been put for u, &c., we proceed to get $\dfrac{dy}{ydx}$, it will give t in a function of x, which will be the integral of Riccati's equation; noticing, that although A and A′ enter the equation, yet its form is such, that they are only equivalent to a single constant.

(13.) Sometimes it will be useful to determine the exponents of the series which represents the dependent variable, particularly when the series is descending; which will readily enable us to determine the series.

Thus, to find the integral of

$$2ax - ay^2 \frac{d^2x}{dy^2} + 2x^2 - 2y^2 \frac{dx^2}{dy^2} = 0.$$

By assuming $\qquad x = \mathrm{A}y^n +, \&c.,$

retaining only the first term, we have

$$\frac{dx}{dy} = n\mathrm{A}y^{n-1} \quad \text{and} \quad \frac{d^2x}{dy^2} = n(n-1)\mathrm{A}y^{n-2},$$

and thence the question reduces to

$$2a\mathrm{A}y^n - n(n-1)a\mathrm{A}y^n + 2\mathrm{A}^2y^{2n} - 2n^2\mathrm{A}^2y^{2n} + = 0,$$

in which the least indices of y are clearly n, while the greatest indices of y are $2n$. Putting the sum of the coefficients of the least power of y = naught, we have $2 - n(n-1) = 0$, whose positive root gives $n = 2$, which enables us to find x in a series of ascending powers of y. For 2 put for n in the equation, reduces it to

$$2a\mathrm{A}y^2 - 2a\mathrm{A}y^2 + 2\mathrm{A}^2y^4 - 8\mathrm{A}^2y^4 = 0;$$

consequently, subtracting the least indices of y from the greatest, we have 2 for their difference, which clearly shows that $$x = Ay^2 + By^4 + Cy^6 +, \&c.,$$ is the proper form for x, when expressed in ascending powers of y. For a descending series, we put the sum of the coefficients of the highest powers of n, in the equation

$$2aAy^n - n(n-1)\,aAy^n + 2A^2y^{2n} - 2n^2A^2y^{2n} = 0,$$

equal to naught, and thence get $1 - n^2 = 0$, whose positive square root is $n = 1$. Putting 1 for n in the equation, we have $2aAy - n(n-1)\,aAy + 2A^2y^2 - 2n^2A^2y^2 = 0$; consequently, subtracting the greatest from the least exponents, we have -1 for their difference, which evidently shows that $x = A'y + B' + C'y^{-1} + D'y^{-2} +, \&c.,$ is the proper form of x in descending powers of y. (See p. 186, &c., of Vince's "Fluxions.")

Taking the first and second differential coefficients of the form $$x = Ay^2 + By^4 +, \&c.,$$ we have $$\frac{dx}{dy} = 2Ay + 4By^3 + 6Cy^5 +, \&c.,$$ and $$\frac{d^2x}{dy^2} = 2A + 12By^2 + 30Cy^4 +, \&c.;$$ consequently, substituting the series for x, and these values of the differential coefficients in the proposed equation, after properly ordering the terms, we shall have

$$\left.\begin{array}{r} 2aAy^2 + 2aBy^4 + 2aCy^6 + 2aDy^8 + \\ -\,2aAy^2 - 12aBy^4 - 30aCy^6 - 56aDy^8 - \\ +\,2A^2y^4 + 4ABy^6 + (2B^2 + 4AC)\,y^8 + \\ -\,8A^2y^4 - 32ABy^6 - (32B^2 + (48AC)\,y^8 - \end{array}\right\} = 0,$$

which must clearly be an identical equation.

Hence, we must have

$$2aA - 2aA = 0, \quad \text{or A is arbitrary};$$

$$2aB - 12aB + 2A^2 - 8A^2 = 0, \quad \text{or} \quad B = -\frac{3A^2}{5a};$$

$$2aC - 30aC + 4AB - 32AB = 0, \quad \text{or} \quad C = \frac{3A^3}{5a^2};$$

$$2aD - 56aD + (2B^2 + 4AC) - (32B^2 + 48AC) = 0,$$

or

$$D = -\frac{31A^4}{45a^3},$$

and so on; consequently, from the substitution of these values of A, B, C, &c., we shall have

$$x = Ay^2 - \frac{3A^2}{5a}y^4 + \frac{3A^3}{5x^2}y^6 - \frac{31A^4}{45a^3}y^8 +, \text{\&c.},$$

for x expressed in a series of ascending powers of y, in which A is the arbitrary constant.

To find x in a series of descending powers of y, we have

$$x = A'y + B' + C'y^{-1} + D'y^{-2} +, \text{\&c.},$$

whose differential coefficients are

$$\frac{dx}{dy} = A' - C'y^{-2} - 2D'y^{-3} -, \text{\&c.},$$

and

$$\frac{d^2x}{dy^2} = 2C'y^{-3} + 6D'y^{-4} +, \text{\&c};$$

consequently, from the substitution of these values in the proposed equation, we have

$$\left.\begin{array}{r} 2aA'y + 2aB' + 2aC'y^{-1} + \text{\&c.} \\ - 2aC'y^{-1} - \text{\&c.} \\ 2A'^2y^2 + 4A'B'y + 2B'^2 + 4B'C'y^{-1} + \text{\&c.} \\ - 2A'^2y^2 + 4A'C' + 4A'D'y^{-1} + \text{\&c.} \\ + 4A'C' + 8A'D'y^{-1} + \text{\&c.} \end{array}\right\} = 0,$$

which must be an identical equation. Hence

$$2A'^2 - 2A'^2 = 0, \quad \text{or} \quad A' \text{ is arbitrary};$$

$$2aA' + 4A'B' = 0, \quad \text{or} \quad B' = -\frac{a}{2};$$

$$2aB' + 2B'^2 + 8A'C' = 0, \quad \text{or} \quad C' = \frac{a^2}{16A'};$$

$$4B'C' + 12A'D' = 0, \quad \text{or} \quad D' = \frac{a^3}{96A'^2},$$

and so on; consequently, from the substitution of these values of A′, B′, C′, &c., we shall have

$$x = A'y - \frac{a}{2} + \frac{a^2}{16A'y} + \frac{a^3}{96A'^2y^2} +, \&c.,$$

for the value of x, when expressed in a series of descending powers of y, in which A′ is the arbitrary constant.

Because the proposed equation is of the second order of differentials, its complete integral must involve two arbitrary constants; consequently, from the addition of the two particular values of x, we get the complete value of

$$x = Ay^2 - \frac{3A^2}{5a}y^4 + \frac{3A^3}{5a^2}y^6 - \&c. + A'y - \frac{a}{2} + \frac{a^2}{16A'y} +, \&c.,$$

as required.

REMARKS.—1. If, with Mr. Young, at p. 260 of his "Integral Calculus," we integrate the equation $\left(1 + \frac{dy}{dx}\right) y = 1$ by the preceding methods, we shall get

$$y = b + \sqrt{2}\,(x-a)^{\frac{1}{2}} - \frac{2}{3}(x-a) + \frac{\sqrt{2}}{18}(x-a)^{\frac{3}{2}} -. \&c.,$$

in which b is the value of y that corresponds to $x = a$, which is clearly equivalent to the determination of the arbitrary constant.

2. This question can clearly be integrated without using series, by regarding x as being a function of y. For the equation can be reduced to

$$y\frac{dy}{dx}=1-y, \quad \text{or} \quad \frac{dx}{dy}=\frac{y}{1-y}=-1+\frac{1}{1-y}=-1-\frac{-1}{1-y},$$

which gives $\qquad dx = -dy - \dfrac{-dy}{1-y};$

and thence

$$x=-y-\log(1-y)=-y+\log\frac{1}{1-y},$$

which needs no correction, supposing y and x to commence together.

(14.) To what has been done, it may be added, that differential equations, of the first order in particular, may often be elegantly integrated in a continued fraction.

Thus, by taking the differential equation

$$P+Qy+Ry^2+S\frac{dy}{dx}=0$$

(See Lacroix, vol. ii., p. 427), and putting

$$y=\frac{Ax^a}{1+y'}, \quad \text{and} \quad Ax^a=X, \quad P'=P+QX+RX^2+S\frac{dX}{dx},$$
$$Q'=2P+QX+S\frac{dX}{dx}, \quad R'=P, \quad S'=-SX,$$

we shall have the transformed equation

$$P'+Q'y'+R'y'^2+S'\frac{dy'}{dx'}=0.$$

If in this equation we put $y'=\dfrac{Bx^b}{1+y''}$, and in the preceding results change $P', Q', R', S',$ and $\dfrac{dy'}{dx},$

into P'', Q'', R'', S'', $\frac{dy''}{dx}$, we shall, in like manner, get

$$P'' + Q''y'' + R''y''^2 + S''\frac{dy''}{dx} = 0$$

for the transformation of the preceding transformed equation, and so on, to any required extent.

To make what has been said more evident, take the particular example

$$my + (1 + x)\frac{dy}{dx} = 0.$$

Then $y = \frac{Ax^a}{1 + y'}$, supposing Ax^a and y' very small, may approximately be reduced to $y = Ax^a$, which gives

$$\frac{dy}{dx} = Aax^{a-1},$$

and thence the proposed equation is approximately reduced to

$$(mA + aA)\,x^a + Aax^{a-1} = 0,$$

which is approximately satisfied by putting $a = 0$, and omitting mA on account of its supposed minuteness; consequently, we may put $y = \frac{A}{1 + y'}$, and shall thence get

$$\frac{dy}{dx} = -\frac{\frac{dy'}{dx}A}{(1 + y')^2}.$$

Hence, from the substitutions of these values of y and $\frac{dy}{dx}$, the proposed equation becomes

$$\frac{mA}{1 + y'} - (1 + x) \times \frac{dy'}{dx}A \div (1 + y')^2 = 0,$$

which is easily reduced to

$$-m - my' + (1 + x)\frac{dy'}{dx} = 0.$$

By changing, as before, y' into Bx^b, and $\frac{dy'}{dx}$ into bBx^{b-1}, this equation becomes

$$-m - mBx^b + bBx^b + bBx^{b-1} = 0;$$

which is clearly satisfied, as required, by putting $b = 1$, and making $B = m$, when terms of the order m^2 are omitted; consequently, $\frac{Bx^b}{1 + y''}$ is reduced to $\frac{mx}{1 + y''}$; noticing, that we have hence reduced y to

$$y = \frac{A}{1 + y'} = \frac{A}{1 + \frac{Bx^a}{1 + y''}} = \frac{A}{1 + \frac{mx}{1 + y''}}.$$

If for y' in the equation

$$-m - my' + (1 + x)\frac{dy'}{dx} = 0,$$

we put its equal, after a slight reduction, we shall get the equation

$$(m - 1)x + [1 + (m - 1)x]y'' + y''^2 + (1 + x)x\frac{dy''}{dx} = 0.$$

Putting Cx^c and cCx^{c-1} for y'' and $\frac{dy''}{dx}$ in this, we get

$$(m - 1)x + [1 + (m - 1)x]Cx^c + C^2x^{2c} + cC(1 + x)x^c = 0;$$

which, by putting $c = 1$, omitting the common factor x, and retaining only the principal terms, reduces to

$$m - 1 + 2C = 0, \quad \text{and} \quad \text{gives} \quad C = -\frac{m-1}{2};$$

consequently, $\frac{Cx^c}{1 + y'''}$ becomes $-\frac{\frac{m-1}{2}}{1 + y'''}$.

Proceeding in this way, we shall get

$$y = \cfrac{A}{1 + \cfrac{mx}{1 - \cfrac{\frac{m-1}{1}\cdot\frac{x}{2}}{1 + \cfrac{\frac{m+1}{3}\cdot\frac{x}{2}}{1 - \cfrac{\frac{m-2}{3}\cdot\frac{x}{2}}{1 + \cfrac{\frac{m+2}{5}\cdot\frac{x}{2}}{1 - \cfrac{\frac{m-3}{5}\cdot\frac{x}{2}}{1 +, \&c.,}}}}}}}$$

for the sought continued fraction, the same as found by Lacroix, at p. 429, vol. ii.

Because the equation

$$my + (1 + x)\frac{dy}{dx} = 0$$

is reducible to $$\frac{dy}{y} = -\frac{mdx}{1 + x},$$

its integral gives $$y = \frac{C}{(1 + x)^m};$$

which gives $y = C$ when $x = 0$, and the continued fraction when $x = 0$ gives $y = A$, and thence $C = A$; consequently, by putting $A = 1$, we shall have

$$\frac{1}{(1 + x)^m} = \cfrac{1}{1 + \cfrac{mx}{1 - \cfrac{\frac{m-1}{1}\cdot\frac{x}{2}}{+, \&c.,}}}$$

or, taking the reciprocals of these equals, we have

$$(1+x)^m = 1 + \cfrac{mx}{1 - \cfrac{\frac{m-1}{1}\cdot\frac{x}{2}}{1 + \cfrac{\frac{m+1}{3}\cdot\frac{x}{2}}{1 - \cfrac{\frac{m-2}{3}\cdot\frac{x}{2}}{1 + \cfrac{\frac{m+2}{5}\cdot\frac{x}{2}}{1 - \cfrac{\frac{m-3}{5}\cdot\frac{x}{2}}{1 +, \&c.;}}}}}}$$

consequently, the binomial theorem may be considered as being reduced to the form of a continued fraction.

Since the exponential theorem (*b*), at page 51, reduces $(1+x)^m$ to

$$1 + m\log(1+x) + \frac{m^2[\log(1+x)]^2}{1.2} +, \&c.,$$

we hence get

$$1 + m\log(1+x) + \&c. = 1 + \cfrac{mx}{1 - \cfrac{\frac{m-1}{1}\cdot\frac{x}{2}}{1 +, \&c.;}}$$

or, from an obvious reduction, we have

$$\log(1+x) + \&c. = \cfrac{x}{1 - \cfrac{\frac{m-1}{1}\cdot\frac{x}{2}}{1 +, \&c.,}}$$

which, by putting $m = 0$, reduces to

$$\log(1+x) = \cfrac{x}{1+\cfrac{\frac{1}{1}\frac{x}{2}}{1+\cfrac{\frac{1}{3}\frac{x}{2}}{1+\cfrac{\frac{2}{3}\frac{x}{2}}{1+\cfrac{\frac{2}{5}\frac{x}{2}}{1+\cfrac{\frac{3}{5}\frac{x}{2}}{1+,\ \&c.;}}}}}}$$

consequently, the hyperbolic logarithm of $1 + x$ is reduced to a continued fraction.

Because $\left(1+\frac{x}{m}\right)^m$, when m is infinite, equals

$$1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \&c. = e^x$$

[see (b') at p. 51], it clearly follows that if in

$$(1+x)^m = 1 + \cfrac{mx}{1-\cfrac{\frac{m-1}{1}.\frac{x}{2}}{1+,\ \&c.,}}$$

we put $\frac{x}{m}$ for x, and suppose m infinite, we shall get

$$e^x = 1 + \cfrac{x}{1 - \cfrac{\frac{1}{1}\cdot\frac{x}{2}}{1 + \cfrac{\frac{1}{3}\cdot\frac{x}{2}}{1 - \cfrac{\frac{1}{3}\cdot\frac{x}{2}}{1 + \cfrac{\frac{1}{5}\cdot\frac{x}{2}}{1 - \cfrac{\frac{1}{5}\cdot\frac{x}{2}}{1 +, \&c.,}}}}}}$$

for the conversion of e^x into a continued fraction.

For another example, we will find the integral of

$$1 - (1 + x^n)\frac{dy}{dx}, \quad \text{or its equivalent} \quad dy = \frac{dx}{1 + x^n}.$$

By taking the integral of

$$dy = \frac{dx}{1 + x^n} = \left(1 - \frac{x^n}{1 + x^n}\right) dx,$$

we have $$y = \int \frac{dx}{1 + x^n} = x - \int \frac{x^n dx}{1 + x^n},$$

which needs no correction, supposing x and y to commence together; and to find y in a continued fraction, we may clearly put $y = \frac{x}{1 + y'}$, which gives

$$\frac{x}{1 + y'} = x - \int \frac{x^n dx}{1 + x^n}, \quad \text{or} \quad \frac{1}{1 + y'} = 1 - \frac{1}{x}\int \frac{x^n dx}{1 + x^n},$$

whose reciprocal gives

$$1 + y' = 1 \div \left(1 - \frac{1}{x}\int \frac{x^n dx}{1 + x^n}\right)$$

$$= 1 + \frac{1}{x}\int \frac{x^n dx}{1 + x^n} \div \left(1 - \frac{1}{x}\int \frac{x^m dx}{1 + x^n}\right),$$

or $$y' = \frac{1}{x}\int\frac{x^n dx}{1+x^n} \div \left(1 - \frac{1}{x}\int\frac{x^n dx}{1+x^n}\right)$$

$$= \left(\frac{x^n}{n+1} - \frac{1}{x}\int\frac{x^{2n}dx}{1+x^n}\right) \div \left(1 - \frac{1}{x}\int\frac{x^n dx}{1+x^n}\right);$$

consequently, by putting $\dfrac{A}{1+y''} = y'$, we thence get

$$\frac{A}{1+y''} = \left(\frac{x^n}{n+1} - \frac{1}{x}\int\frac{x^{2n}dx}{1+x^n}\right) \div \left(1 - \frac{1}{x}\int\frac{x^n dx}{1+x^n}\right),$$

and thence $A = \dfrac{x^n}{n+1}$ is the numerator of the second of the continued fractions, and

$$\frac{1}{1+y''} = \left(1 - \frac{n+1}{x^{n+1}}\int\frac{x^{2n}dx}{1+x^n}\right) \div \left(1 - \frac{1}{x}\int\frac{x^n dx}{1+x^n}\right);$$

or, taking its reciprocal, we have

$$1 + y'' = \left(1 - \frac{1}{x}\int\frac{x^n dx}{1+x^n}\right) \div \left(1 - \frac{n+1}{x^{n+1}}\int\frac{x^{2n}dx}{1+x^n}\right)$$

$$= 1 + \left(\frac{n+1}{x^{n+1}}\int\frac{x^{2n}dx}{1+x^n} - \frac{1}{x}\int\frac{x^n dx}{1+x^n}\right) \div \left(1 - \frac{n+1}{x^{n+1}}\int\frac{x^{2n}dx}{1+x^n}\right),$$

or, after a slight reduction,

$$y'' = \left(\frac{(n+1)x^n}{2n+1} - \frac{x^n}{n+1} - \frac{n+1}{x^{n+1}}\int\frac{x^{3n}dx}{1+x^n} + \frac{1}{x}\int\frac{x^{2n}dx}{1+x^n}\right)$$

$$\div \left(1 - \frac{n+1}{x^{n+1}}\int\frac{x^{2n}dx}{1+x^n}\right)$$

$$= \left(\frac{n^2x^n}{(n+1)(2n+1)} + \frac{1}{x}\int\frac{x^{2n}dx}{1+x^n} - \frac{n+1}{x^{n+1}}\int\frac{x^{3n}dx}{1+x^n}\right)$$

$$\div \left(1 - \frac{n+1}{x^{n+1}}\int\frac{x^{2n}dx}{1+x^n}\right).$$

Putting $y'' = \dfrac{B}{1+y'''}$, we shall easily get

$$B = \frac{n^2x^n}{(n+1)(2n+1)}$$

for the numerator of the third of the continued fractions; and then taking the reciprocal, we shall have

$$1 + y''' = \left(1 - \frac{n+1}{x^{n+1}} \int \frac{x^{2n}dx}{1+x^n}\right)$$

$$\div \left(1 + \frac{(n+1)(2n+1)}{n^2 x^{n+1}} \int \frac{x^{2n}dx}{1+x^n} - \frac{(n+1)^2(2n+1)}{n^2 x^{2n+1}} \int \frac{x^{3n}dx}{1+x^n}\right),$$

and so on, to any required extent. Hence, we shall have

$$y = \int \frac{dx}{1+x^n} = \cfrac{x}{1 + \cfrac{\cfrac{x^n}{n+1}}{1 + \cfrac{\cfrac{n^2 x^n}{(n+1)(2n+1)}}{1 + \cfrac{\cfrac{(n+1)^2 x^n}{(2n+1)(3n+1)}}{1 + \cfrac{\cfrac{(2n)^2 x^n}{(3n+1)(4n+1)}}{1 +, \&c.,}}}}}$$

for the sought continued fraction. (See Lacroix, vol. ii., p. 431.)

If $n = 1$, this formula gives the same expression for $\log(1 + x)$, as at p. 560; and if $n = 2$, we shall have

$$\int \frac{dx}{1+x^2} = \tan^{-1} x = \cfrac{x}{1 + \cfrac{\cfrac{x^2}{1.3}}{1 + \cfrac{\cfrac{4x^2}{3.5}}{1 + \cfrac{\cfrac{9x^2}{5.7}}{1 + \cfrac{\cfrac{16x^2}{7.9}}{1 +, \&c.}}}}}$$

(15.) We will now proceed to show how to find the integrals of what are called *Simultaneous Equations*, such as

$$My + Nx + P\frac{dy}{dt} + Q\frac{dx}{dt} = T,$$

and
$$M'y + N'x + P'\frac{dy}{dt} + Q'\frac{dx}{dt} = T',$$

in which M, N, P, &c., are supposed to be functions of t, considered as being the independent variable; the equations being coexistent and of analogous forms.

1. To integrate this kind of equations, after multiplying by the differential of the independent variable dt, they may be written in the forms

$$(My + Nx)\,dt + Pdy + Qdx = Tdt,$$

and
$$(M'y + N'x)\,dt + P'dy + Q'dx = T'dt;$$

and multiplying the second of these by θ, regarded as being a function of t, and adding the product to the first, we get the single equation

$$[(M + M'\theta)\,y + (N + N'\theta)\,x]\,dt + (P + P'\theta)\,dy + (Q + Q'\theta)\,dx = (T + T'\theta)\,dt.$$

Putting

$$M + M'\theta = M_1,\quad N + N'\theta = N_1,\quad P + P'\theta = P_1,$$
$$Q + Q'\theta = Q_1,\quad T + T'\theta = T_1,$$

we thence have

$$(M_1y + N_1x)\,dt + P_1dy + Q_1dx = T_1dt,$$

a form analogous to the proposed equations, and it is clear that this equation is equivalent to

$$M_1\left(y + \frac{N_1}{M_1}x\right)dt + P_1\left(dy + \frac{Q_1}{P_1}dx\right) = T_1dt.$$

Putting $$y + \frac{N_1}{M_1} x = z$$

and assuming $$d\left(y + \frac{N_1}{M_1} x\right) = dy + \frac{Q_1}{P_1} dx,$$

the preceding equation is reduced to the form

$$M_1 z dt + P_1 dz = T_1 dt, \quad \text{or} \quad dz + \frac{M_1}{P_1} z dt = \frac{T_1}{P_1} dt,$$

which (see p. 455) is a linear equation. From

$$d\left(y + \frac{N_1}{M_1} x\right) = dy + \frac{Q_1}{P_1} dx,$$

by taking the indicated differentials, we have

$$dy + \frac{N_1}{M_1} dx + x d \frac{N_1}{M_1} = dy + \frac{Q_1}{P_1} dx,$$

or $$\frac{N_1}{M_1} dx + x.l \frac{N_1}{M_1} = \frac{Q_1}{P_1} dx,$$

which must be an identical equation; consequently,

$$\frac{N_1}{M_1} = \frac{Q_1}{P_1} \quad \text{and} \quad d \frac{N_1}{M_1} = 0,$$

or $$\frac{N + N'\theta}{M + M'\theta} = \frac{Q + Q'\theta}{P + P'\theta} \quad \text{and} \quad d \frac{N + N'\theta}{M + M'\theta} = 0,$$

and by performing the indicated differentiation of the second of these equations, and eliminating θ from the result and the preceding equation, we shall get the relation between the coefficients of the proposed equations that must exist, in order that their integration may be reduced to the integration of the preceding linear differential equation of the first order. It may be noticed in this place that, if we integrate the equation

$d \dfrac{N + N'\theta}{M + M'\theta} = 0$, we shall have $\dfrac{N + N'\theta}{M + M'\theta} = C =$ a constant, and thence $$\frac{N + N'\theta}{M + M'\theta} = \frac{Q + Q'\theta}{P + P'\theta}$$

reduces to $$\frac{Q + Q'\theta}{P + P'\theta} = C;$$

consequently, eliminating θ from these equations, we have

$$\theta = \frac{N - CM}{CM' - N'} \quad \text{and} \quad \theta = \frac{Q - CP}{CP' + Q'},$$

which give $$\frac{N - CM}{CM' - N'} = \frac{Q - CP}{CP' - Q'},$$

or, reducing this to a common denominator, &c., we have

$$C^2 + \frac{NP' - PN + MQ' - M'Q}{PM' - MP'} C = \frac{NQ' - QN'}{PM' - MP'}.$$

Hence, having found C from the solution of this quadratic, and taken the differential of its value on the supposition of the constancy of C, we shall clearly get the same result as from the preceding method.

Solving the linear equation will clearly give z in terms of t; and thence from $y + \frac{N'}{M'} x = z$, we can find y in terms of x and t, which, being substituted in either of the proposed equations, will give a differential equation in x and t, whose integral gives x in t, and thence having found x and y in terms of t, by eliminating t we shall get y in terms of x, as required.

2. If the coefficients M, N, P, &c., in the first members of the proposed equations, are all constant, it is clear that

$$d\,\frac{N + N\theta}{M + M'} = 0$$

is satisfied by supposing that θ is constant; and thence, from the solution of the equation

$$\frac{N + N'\theta}{M + M'\theta} = \frac{Q + Q'\theta}{P + P'\theta},$$

we shall get, by the solution of a quadratic equation, two constant values, θ' and θ'', of θ; consequently, if m and n are the coefficients of the linear equation that correspond to θ', and m' and n' are those that correspond to θ'', the linear equation gives the two linear equations

$$dz + mzdt = ndt \quad \text{and} \quad dz + m'zdt = n'dt.$$

Integrating these equations by the formula at p. 455, we shall have

$$z = e^{-\int mdt}\left(\int ne^{\int mdt}dt\right) \quad \text{and} \quad z = e^{-\int m'dt}\left(\int n'e^{\int m'dt}dt\right)$$

for the sought integrals; noticing, that the arbitrary constants are supposed to be comprehended by the integral signs $\int n$, &c., $\int n'$, &c. By substituting the values of $y + \frac{N'}{M'}x$ that correspond to those different values of z, for z in the preceding integrals, we shall have two equations in x, y, and t, which will clearly give x and y in terms of t; consequently, from the elimination of t, we shall get y in terms of x.

3. For another example, we will integrate the simultaneous equations

$$dy + (My + Nx + Pz)\,dt = Tdt,$$

$$dx + (M'y + N'x + P'z)\,dt = T'dt,$$

$$dz + (M''y + N''x + P''z)\,dt = T''dt;$$

which may be supposed to be obtained from three equations of forms analogous to those of the preceding example, by eliminating $\frac{dx}{dt}$ and $\frac{dz}{dt}$ from the first by means of the second and third equations, and so on for the remaining equations.

Supposing T, T′, T″, to be functions of t, while the other coefficients in the preceding equations are constant; then, multiplying the second and third equations by the constants C and C′, and adding the products and the first together, we shall have a single equation of the form

$$dy + \mathrm{C}dx + \mathrm{C}'dz + \mathrm{Q}\,(y + \mathrm{R}x + \mathrm{S}z)\,dt = \mathrm{U}dt.$$

If in this we change C and C′ into R and S, it will become

$$dy + \mathrm{R}dx + \mathrm{S}dz + \mathrm{Q}\,(y + \mathrm{R}x + \mathrm{S}z)\,dt = \mathrm{U}dt;$$

consequently, putting $y + \mathrm{R}x + \mathrm{S}z = v$,

since R and S are constants, the equation will become

$$dv + \mathrm{Q}vdt = \mathrm{U}dt,$$

a linear equation, whose integral gives v, or its equal

$$y + \mathrm{R}x + \mathrm{S}z,$$

equal to a function of t.

4. The preceding process is applicable to differential equations of the higher orders, which may clearly be reduced to those of the first order.

Thus, the equations

$$d^2y + (\mathrm{M}y + \mathrm{N}x)\,dt^2 + (\mathrm{P}dy + \mathrm{Q}dx)\,dt = \mathrm{T}dt^2$$

and $d^2x + (\mathrm{M}'y + \mathrm{N}'x)\,dt^2 + (\mathrm{P}'dy + \mathrm{Q}'dx)\,dt = \mathrm{T}'dt^2,$

by putting $dy = pdt$ and $dx = qdt$,

are reduced to the equations

$$dy = pdt, \quad dx = qdt,$$

$$dp + (\mathrm{M}y + \mathrm{N}x + \mathrm{P}p + \mathrm{Q}q)\,dt = \mathrm{T}dt,$$

and $dq + (\mathrm{M}'y + \mathrm{N}'x + \mathrm{P}'p + \mathrm{Q}'q)\,dt = \mathrm{T}'dt,$

to which the preceding method can evidently be applied. (See Young, p. 264, &c.; and Lacroix, p. 337, &c.)

By reducing the first two of the preceding equations to

$$dy - pdt = 0, \quad dx - qdt = 0,$$

and multiplying the second, third, and fourth by the constants C, C′, C″, and adding the products, we have the single equation

$$dy + \mathrm{C}dx + \mathrm{C}'dp + \mathrm{C}''dq + \mathrm{Q}(y + \mathrm{R}x + \mathrm{S}p + \mathrm{V}q)\,dt = \mathrm{U}dt.$$

Putting

$$dy + \mathrm{C}x + \mathrm{C}'p + \mathrm{C}''q = d(y + \mathrm{R}x + \mathrm{S}p + \mathrm{V}q),$$

and $\mathrm{C} = \mathrm{R}, \quad \mathrm{C}' = \mathrm{S}, \quad \mathrm{C}'' = \mathrm{V};$

then, since these values are constant, our equation is reduced to the linear form $dv + \mathrm{Q}vdt = \mathrm{U}dt$,

in which v is put for

$$y + \mathrm{R}x + \mathrm{S}p + \mathrm{V}q,$$

and thence, by taking the integral, this becomes a function of t. For a simpler method of integrating simultaneous equations of the second order, under certain restrictions, we shall refer to p. 130 of Whewell's "Dynamics," or to any other work that treats of the very small vibrations of what are called *Complex Pendulums.*

SECTION IX.

INTEGRATION OF DIFFERENTIAL EQUATIONS CONTAINING THREE VARIABLES.

(1.) If we have $\mathrm{P}dx + \mathrm{Q}dy + \mathrm{R}dz = 0$,
such, that x and y are considered as independent variables, and z a function of them, then, if $p = -\frac{\mathrm{P}}{\mathrm{R}}$ and $q = -\frac{\mathrm{Q}}{\mathrm{R}}$, the equation will be reduced to the form $dz = pdx + qdy$.

If this is the *total differential* of z, regarded as being a function of x and y, it is evident that we shall have

$$p = \frac{dz}{dx} \quad \text{and} \quad q = \frac{dz}{dy},$$

and because dz is supposed to be an exact differential, its equivalent $pdx + qdy$ must also be an exact differential; consequently, Euler's condition of integrability (see pp. 439 and 440) must exist, which gives the differential coefficient of p taken relatively to y equal to the differential coefficient of q taken relatively to x, and thence, since p and q may contain z, we shall get

$$\frac{dp}{dy} + \frac{dp}{dz}\,\frac{dz}{dy} = \frac{dq}{dx} + \frac{dq}{dz}\,\frac{dz}{dx} \quad \text{or} \quad \frac{dp}{dy} + q\,\frac{dp}{dz} = \frac{dq}{dx} + p\,\frac{dq}{dz},$$

or, by transposition, we have

$$\frac{dp}{dy} - \frac{dq}{dx} + q\,\frac{dp}{dz} - p\,\frac{dq}{dz} = 0$$

for the condition of integrability of $dz = pdx + qdy$.

Because we have supposed that

$$p = -\frac{P}{R} \quad \text{and} \quad q = -\frac{Q}{R},$$

we thence get

$$-\frac{dp}{dy} = \frac{R\frac{dP}{dy} - P\frac{dR}{dy}}{R^2}, \quad -\frac{dq}{dx} = \frac{R\frac{dQ}{dx} - Q\frac{dR}{dx}}{R^2},$$

$$q\frac{dp}{dz} = \frac{Q}{R}\,\frac{R\frac{dP}{dz} - P\frac{dR}{dz}}{R^2}, \quad p\frac{dq}{dz} = \frac{P}{R}\,\frac{R\frac{dQ}{dz} - Q\frac{dR}{dz}}{R^2};$$

consequently, from the substitution of $\frac{dp}{dy}$, $\frac{dq}{dx}$, &c., in the preceding equation, it becomes

$$P\frac{dR}{dy} - R\frac{dP}{dy} + R\frac{dQ}{dx} - Q\frac{dR}{dx} + Q\frac{dP}{dz} - P\frac{dQ}{dz} = 0,$$

which expresses the condition of integrability of the equation

$$Pdx + Qdy + Rdz = 0,$$

on the supposition that when multiplied by the factor $\frac{1}{R}$, it is reduced to

$$dz + \frac{P}{R}dx + \frac{Q}{R}dy = 0,$$

or its equivalent

$$dz = -\frac{P}{R}dx - \frac{Q}{R}dy = pdx + qdy,$$

which, by supposition, is an exact differential equation.

Hence, to find the integral of the differential equation

$$Pdx + Qdy + Rdz = 0,$$

we examine it to see if the preceding condition of integrability is satisfied; then, if it is satisfied, we multiply it by some factor m, which reduces it to the form

$$mPdx + mQdy + mRdz = 0.$$

To determine the proper form of m, we may omit any one of the terms of the equation, as the last, then we find m such that
$$mPdx + mQdy = 0$$
shall be an exact differential, on the supposition of the constancy of z; and putting
$$du = mPdx + mQdy,$$
by taking the integral, we have
$$u = \int (mPdx + mQdy) + \phi(z) = V + \phi(z);$$
in which $\phi(z)$ = a function of z, is used for the arbitrary constant, since, in the integration, z has been considered as a constant.

To find $\phi(z)$, we differentiate the members of the equation
$$u = V + \phi(z),$$
relatively to z only, and thence get
$$\frac{du}{dz} = \frac{dV}{dz} + \frac{d\phi(z)}{dz};$$
consequently, since u is here supposed to be the integral of
$$mPdx + mQdy + mRdz = 0, \quad \frac{du}{dz} = mR,$$
and thence
$$mR = \frac{dV}{dz} + \frac{d\phi(z)}{dz}, \quad \text{or} \quad \frac{d\phi(z)}{dz} = mR - \frac{dV}{dz},$$
which gives
$$\phi(z) = \int \left(mR - \frac{dV}{dz}\right) dz,$$
and thence the integral becomes known.

Because $\phi(z)$ is independent of either x or y, it is clear that when the factor m is correctly found, it must be independent of either x or y.

Thus, to integrate $yzdx - xzdy + yxdz = 0.$

We have $P = yz$, $Q = -xz$, and $R = yx$,

and thence the equation of condition becomes

$$P\frac{dR}{dy} - R\frac{dP}{dy} + R\frac{dQ}{dx} - Q\frac{dR}{dx} + Q\frac{dP}{dz} - P\frac{dQ}{dz} =$$

$$yzx - yxz - yxz + xzy - xzy + yzx = 0,$$

and the condition being satisfied, the proposed equation must be integrable.

To find the integral, we omit the last term, and thence get

$$z(ydx - xdy) = 0,$$

which becomes an exact integral, by multiplying it by the factor $m = \frac{1}{y^2}$, the integral being $u = \frac{zx}{y} + \phi(z)$;

consequently, $\qquad \frac{du}{dz} = \frac{z}{y} + \frac{d\phi(z)}{dz}$;

or, since $\qquad \frac{du}{dz} = \frac{yx}{y^2} = \frac{x}{y}$,

we shall get $\qquad \frac{x}{y} = \frac{x}{y} + \frac{d\phi(z)}{dz}$,

which gives $\qquad d\phi(z) = 0$ and $\phi(z) = \text{const.} = C$,

and thence $u = \frac{zx}{y} + C$ is the sought integral.

For another example, we may take the equation

$$zydx + xzdy + xydz + az^2dz = 0,$$

which will be found to satisfy the condition of integrability; consequently, its integral can be found. Indeed, since the integral of the first three terms of the equation is xyz, and that $\frac{az^3}{3}$ is the integral of the fourth term, it is clear that the integral of the equation is

$$u = xyz + \frac{az^3}{3} + C = 0,$$

in which C represents the arbitrary constant.

(2.) If the equation $Pdx + Qdy + Rdz = 0$ does not satisfy the condition of integrability, then it is clear that one of the variables can not be regarded as being a function of the other two, so that the variables can not represent a surface; yet, as shown by Monge, they may represent a pair of integrals which depend on an arbitrary function of z considered as being the dependent variable.

For, as shown at p. 513, &c., by regarding the dependent variable z as being constant, the resulting equation

$$Pdx + Qdy = 0,$$

admits of an integral. Hence, multiplying

$$Pdx + Qdy + Rdz = 0$$

by m, as before, so as to make $(Pdx + Qdy)\, m = 0$ an exact differential, we shall have

$$Pmdx + Qmdy + Rmdz = 0.$$

Putting $$du = Pmdx + Qmdy,$$

we shall, as before, get

$$u = \int (Pmdx + Qmdy) = V + \phi(z) = 0$$

for one of the integrals; and taking the differential coefficient of this relative to z, we get $\frac{dV}{dz} + \frac{d\phi(z)}{dz}$, which, being put equal to Rm, the coefficient of dz, in the equation

$$Pmdx + Qmdy + Rmdz,$$

gives $$\frac{dV}{dz} + \frac{d\phi(z)}{dz} = Rm$$

for the other equation; consequently

$$V + \phi(z) = 0 \quad \text{and} \quad \frac{dV}{dz} + \frac{d\phi(z)}{dz} = Rm,$$

in which $\phi(z)$ is an arbitrary function of z, which satisfy the equation $\qquad Pmdx + Qmdy + Rmdz = 0.$

Thus, of $\qquad ydy + zdx - dz = 0,$

regarding z as invariable, the multiplier m is 2, and thus the equation to be integrated becomes

$$2ydy + 2zdx - 2dz = 0;$$

the integral of its first two terms, regarding z as const., is

$$y^2 + 2zx + \phi(z) = 0,$$

and thence $\qquad \dfrac{dV}{dz} + \dfrac{d\phi(z)}{dz} = Rm$

becomes $\; 2x + \dfrac{d\phi(z)}{dz} = -2, \;$ or $\; 2x + \dfrac{d\phi(z)}{dz} + 2 = 0,$

which, by putting $\phi(z) = z^2$, is immediately reduced to $x + z + 1 = 0$, the equation of a right line.

In much the same way

$$2xzdx + 2yzdy + x^2dz = 0$$

can be satisfied by

$$(x^2 + y^2)z + \phi(z) = 0 \quad \text{and} \quad x^2 + y^2 + \frac{d\phi(z)}{dz} = x^2,$$

or $\qquad y^2 + \dfrac{d\phi(z)}{dz} = 0.$

For another example, we will take the equation

$$\frac{xdx + ydy}{x(x-a) + y(y-b)} - \frac{dz}{z-c} = 0.$$

This equation can be immediately satisfied by putting

$$x(x-a)+y(y-b)=\phi(z)=\text{a function of } z,$$

which reduces the proposed equation to

$$2xdx+2ydy=\frac{2\phi(z)}{z-c}dz,$$

whose integral is $\quad x^2+y^2=2\int\frac{\phi(z)}{z-c}dz.$

It is easy to perceive that, by putting $\phi(z)=-(z-c)z$, the integral becomes

$$x^2+y^2=-z^2+R^2, \quad \text{or} \quad x^2+y^2+z^2=R^2,$$

the equation of a spheric surface, in which R^2 is used for the constant.

It may be added, that, if we put $y=x$, the differential is immediately reduced to $\dfrac{dz}{z-c}=\dfrac{2dx}{2x-a-b}$, whose integral is clearly $\quad z-c=C(2x-a-b).$

(3.) It may be observed that the differentials and their integrals here considered being of algebraic forms, their integrals are sometimes called *algebraic integrals.*

Algebraic integrals of differential equations can sometimes be obtained from the simplest principles.

Thus, to find the algebraic integral of

$$dx\sqrt{1+x^3}+dy\sqrt{1+y^3}=0,$$

or of its equivalent

$$dx(1+x)\sqrt{\frac{x^2-x+1}{x+1}}+dy(1+y)\sqrt{\frac{y^2-y+1}{y+1}}=0,$$

we may proceed as follows.

By assuming

$$\frac{x^2 - x + 1}{x + 1} = v \quad \text{or} \quad x^2 - (1 + v)\,x + 1 - v = 0,$$

and using x and y to represent its roots, we shall, from the well-known theory of equations, get

$$x + y = 1 + v \quad \text{and} \quad xy = 1 - v,$$

whose sum gives $\qquad x + y + xy = 2$

for an algebraic integral of the equation. Because x and y are roots of $\quad x^2 - (1 + v)\,x + 1 - v = 0,$

it is clear that we shall have

$$\sqrt{\frac{x^2 - x + 1}{x + 1}} = \sqrt{\frac{y^2 - y + 1}{y + 1}} = v;$$

consequently, by erasing this common factor from the second form of the proposed equation, we shall get the differential

equation $\qquad (1 + x)\,dx + (1 + y)\,dy = 0,$

whose integral is

$$x + y + \frac{x^2 + y^2}{2} = \mathrm{C} = \text{the arbitrary constant.}$$

From the algebraic equation we have $x + y = 2 - xy$, which, substituted in

$$x + y + \frac{x^2 + y^2}{2} = \mathrm{C},$$

reduces it to

$$2 - xy + \frac{x^2 + y^2}{2} = \mathrm{C}, \quad \text{or} \quad \frac{(x - y)^2}{2} = \mathrm{C} - 2,$$

or, more simply,

$$(x - y)^2 = \mathrm{C}' = \text{constant},$$

an integral that is evidently of an algebraic form.

REMARK.—If we proceed in like manner to integrate

$$\frac{dx}{\sqrt{1 + x^3}} + \frac{dy}{\sqrt{1 + y^3}} = 0,$$

we shall get $x^2 - (1 + v)\, x + 1 - v = 0,$

and thence $x + y + xy = 2$

for the algebraic integral, the same as before. Hence, from

$$\sqrt{\frac{x + 1}{x^2 - x + 1}} = \sqrt{\frac{y + 1}{y^2 - y + 1}},$$

the proposed differential equation reduces to

$$\frac{dx}{1 + x} + \frac{dy}{1 + y} = 0, \quad \text{or to} \quad (1 + y)\, dx + (1 + x)\, dy = 0,$$

whose integral is

$$x + y + xy = C = \text{the arbitrary constant},$$

which, by putting 2 for C, becomes $x + y + xy = 2$, which is the same as the preceding algebraic integral.

For another example, we will show how to find the algebraic integrals of

$$dx\, \sqrt{1 + x^3} + dy\, \sqrt{1 + y^3} + dz\, \sqrt{1 + z^3} = 0.$$

Because

$$dx\, \sqrt{1 + x^3} = dx\, (x^2 + x) \sqrt{\frac{1 + x^3}{(x + x^2)^2}}$$

$$= dx\, (x^2 + x) \sqrt{\frac{x^2 - x + 1}{x^3 + x^2}}, \text{ \&c.,}$$

by putting $\frac{x^3 + x^2}{x^2 - x + 1} = v,$

we have $x^3 + (1 - v)\, x^2 + vx - v = 0;$

and supposing x, y, and z, to be its roots, we shall have

$x + y + z = v - 1, \quad xy + xz + yz = v, \quad \text{and} \quad xyz = v;$

consequently, eliminating v from these equations, we shall have

$xy + xz + yz - (x + y + z) = 1$ and $xy + xz + yz = xyz$,

which clearly correspond to two of the algebraic equations. To get the other algebraic equation, we reject the factor

$$\sqrt{\frac{x^2 - x + 1}{x^3 + x^2}} = \sqrt{\frac{y^2 - y + 1}{y^3 + y^2}} = \sqrt{\frac{z^2 - z + 1}{z^3 + z^2}} = \frac{1}{\sqrt{v}},$$

which is common to all the terms of the proposed equation, and thence get the differential equation

$$(x^2 + x)\,dx + (y^2 + y)\,dy + (z^2 + z)\,dz = 0;$$

and by taking the integral of this, we have

$$\frac{x^3}{3} + \frac{x^2}{2} + \frac{y^3}{3} + \frac{y^2}{2} + \frac{z^3}{3} + \frac{z^2}{2} = \text{constant},$$

or $\quad 2(x^3 + y^3 + z^3) + 3(x^2 + y^2 + z^2) = C,$

which is clearly an algebraic equation, as required.

For the last example of this method of finding algebraic integrals, we will take

$$dx\sqrt{1 + x^4} + dy\sqrt{1 + y^4} = 0.$$

By putting $x^2 = x'$ and $y^2 = y'$, we shall change the equation to

$$\frac{1}{2}\,dx'\sqrt{\frac{1 + x'^2}{x'}} + \frac{1}{2}\,dy'\sqrt{\frac{1 + y'^2}{y'}}.$$

Putting

$$\frac{1 + x'^2}{x'} = \frac{1 + y'^2}{y'} = v, \quad \text{we get} \quad x'^2 - vx' + 1 = 0,$$

whose roots being x' and y', we have

$x' + y' = v$, or $x^2 + y^2 = v$, and $x'y' = 1$, or $x^2y^2 = 1$.

Rejecting the factor $\frac{v}{2}$ from the proposed equation, we get

$$dx' + dy' = 0,$$

whose integral gives $x' + y' = x^2 + y^2 = C$;

and thence $x^2y^2 = 1, \quad x^2 + y^2 = C,$

are algebraic integrals of the proposed equation.

For fuller information on algebraic integrals, see pages 383–404 of Professor Gill's "Mathematical Miscellany," published at Flushing, L. I., during 1836, 1837, &c.; and for other methods of finding algebraic integrals, together with their applications to elliptic functions, see the "Exercices de Calcul Intégral," of Legendre, and p. 471, &c., of Lacroix.

(4.) Resuming $dz = \frac{dz}{dx}dx + \frac{dz}{dy}dy = pdx + qdy,$

in which z is a function of x and y, considered as being independent variables, so that dx and dy (see p. 34) must be constant in differentiating the equation; consequently, by taking the differential of the equation, we shall have

$$d^2z = \frac{d^2z}{dx^2}dx^2 + 2\frac{d^2z}{dxdy}dxdy + \frac{d^2z}{dy^2}dy^2;$$

or, representing $\frac{d^2z}{dx^2}$, $\frac{d^2z}{dxdy} = \frac{d^2z}{dydx}$, $\frac{d^2z}{dy^2}$, by r, s, and t,

it becomes $d^2z = rdx^2 + 2sdxdy + tdy^2$;

and from $\frac{dz}{dx} = p$ and $\frac{dz}{dy} = q$

we also have $d\frac{dz}{dx} = \frac{d^2z}{dx^2}dx + \frac{d^2z}{dxdy}dy$

and $d\frac{dz}{dy} = \frac{d^2z}{dydx}dx + \frac{d^2z}{dy^2}dy,$

or their equivalents

$$dp = rdx + sdy \quad \text{and} \quad dq = sdx + tdy.$$

If we differentiate the equation

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = R^2$$

successively, according to the preceding principles, we shall get

$$(x - a)\,dx + (y - b)\,dy + (z - c)\,dz = 0,$$

$$dx^2 + dy^2 + dz^2 + (z - c)\,d^2z = 0,$$

and

$$3dzd^2z + (z - c)\,d^3z = 0,$$

for its first, second, and third differentials; and so on, to any required extent.

It is evident that the preceding forms will be very useful in finding, by reverse processes, the second, third, &c., integrals of differential equations between x, y, and z, when z is a function of x and y regarded as being independent variables.

SECTION X.

PARTIAL DIFFERENTIAL EQUATIONS.

(1.) INTEGRATION of partial differential equations of the first order between x, y, and z, z being considered as being a function of x and y, regarded as being independent variables.

A partial differential equation between x, y, and z, is said to be of the first order, when it involves $\frac{dz}{dx}$ or $\frac{dz}{dy}$, or both of these differential coefficients, together with constants and one or more of the variables, according to the nature of the case. It is hence clear that a partial differential coefficient can not exist between only two variables, as x and y; since if one of them, as y, is a function of the other, the coefficient $\frac{dy}{dx}$ must evidently be *complete* or *total*, and not partial or incomplete.

(2.) The simplest partial differential coefficient that can exist between z, and x, y, must evidently be of the form $\frac{dz}{dx} = a$, obtained by regarding y as constant in the differentiation; consequently, reversing the process, in the integration, we multiply by dx, and thence get $dz = adx$, whose integral gives $z = ax + \phi y$ by using an arbitrary function of y to complete the integral; since y was regarded as constant in obtaining the proposed differential coefficient.

In like manner, from $\frac{dz}{dx} = \text{Y}$, a function of y, we get $z = \text{Y}x + \phi y$; and from $\frac{dz}{dx} = \text{X} =$ a function of x, we have $z = \int \text{X}dx + \phi y$.

EXAMPLES.

1. The integral of $\frac{dz}{dx} = x^2 + yx + a$ is required.

Multiplying by dx and integrating, since y and x are independent variables, clearly gives

$$z = \frac{x^3}{3} + y\frac{x^2}{2} + ax + \phi y.$$

2. To integrate

$$\frac{dz}{dx} = \frac{2x}{y^2 + x^2} \quad \text{and} \quad \frac{dz}{dy} = \frac{y}{\sqrt{(x^2 + y^2)}}.$$

The answers are

$$z = \log (y^2 + x^2) + \phi y \quad \text{and} \quad z = \sqrt{x^2 + y^2} + \phi x.$$

3. To integrate

$$\frac{dz}{dx} = \frac{1}{\sqrt{(y^2 - x^2)}} \quad \text{and} \quad \frac{dz}{dy} = \frac{1}{\sqrt{(x^2 + y^2)}}.$$

The answers are

$$z = \sin^{-1}\frac{x}{y} + \phi y \quad \text{and} \quad z = \log [\sqrt{(x^2 + y^2)} + y] + \phi x.$$

4. The integrals of

$$\frac{dz}{dx} = f(x, y) \quad \text{and} \quad \frac{dz}{dx} = \pm \frac{dz}{dy} = \text{a function of } x \text{ and } y$$

are required.

The answers are

$$z = \int f(x, y)\, dx + \phi y \quad \text{and} \quad z = \pm \int \frac{dz}{dy}\, dx + \phi y.$$

(3.) We now propose to show how to integrate the equation

$$M\left(\frac{dz}{dx}\right) + N\left(\frac{dz}{dy}\right) = 0,$$

on the supposition that M and N are functions of x and y.

From the equation we readily get

$$\frac{dz}{dy} = -\frac{M}{N}\left(\frac{dz}{dx}\right),$$

which, being substituted for $\frac{dz}{dy}$ in

$$dz = \frac{dz}{dx}\, dx + \frac{dz}{dy}\, dy,$$

that results from the consideration that z is a function of the independent variables x and y, gives

$$dz = \frac{dz}{dx}\left(dx - \frac{M}{N}\, dy\right) = \frac{dz}{dx}\, \frac{N dx - M dy}{N}.$$

From what is shown at p. 513, since $N dx - M dy$ is a differential between x and y, it clearly admits of a factor l, which makes it an exact differential, denoted by du; or, more generally, $\frac{N dx - M dy}{N}$, being a differential between x and y, admits of a factor m, which makes it an exact differential, denoted by dv; consequently, we shall have

$$l\,(N dx - M dy) = du, \quad \text{or} \quad m\left(\frac{N dx - M dy}{N}\right) = dv.$$

Hence, eliminating

$$N dx - M dy, \quad \text{or} \quad \frac{N dx - M dy}{N},$$

from the value of dz, it will be reduced to

$$dz = \frac{1}{l\mathrm{N}}\frac{dz}{dx}\,du, \quad \text{or to} \quad dz = \frac{1}{m}\frac{dz}{dx}\,dv.$$

Hence, since $\frac{dz}{dx}$ is arbitrary, we may clearly suppose it to be so taken that

$$dz = \frac{1}{l\mathrm{N}}\frac{dz}{dx}\,du$$

may be exactly integrable, and of course

$$\frac{1}{l\mathrm{N}}\frac{dz}{dx} = \mathrm{F}u = \text{a function of } u,$$

and thence $z = \phi u$; and, in like manner, from

$$dz = \frac{1}{m}\frac{dz}{dx}\,dv$$

we shall get $z = \psi v$, a function of v, which must clearly be the same as the preceding function.

Thus, to integrate

$$x\left(\frac{dz}{dy}\right) - y\left(\frac{dz}{dx}\right) = 0,$$

by comparing it to the general formula we get

$$\mathrm{M} = -y \quad \text{and} \quad \mathrm{N} = x,$$

and thence

$$l(\mathrm{N}dx - \mathrm{M}dy) = du \quad \text{becomes} \quad l(xdx + ydy) = du,$$

which gives $\quad l = 2 \quad$ and $\quad u = x^2 + y^2$;

consequently, $\qquad z = \phi(x^2 + y^2).$

Similarly, from $m\left(\frac{\mathrm{N}dx - \mathrm{M}dy}{\mathrm{N}}\right) = dv,$

we get

$$m\left(\frac{xdx + ydy}{x}\right) = dv,$$

which gives $m = 2x$ for the sought factor, and thence

$$v = x^2 + y^2;$$

consequently, we thus get $z = \psi(x^2 + y^2)$, which is essentially the same as the preceding result; and from what is shown at p. 215, $z = \phi(x^2 + y^2)$ becomes the general equation of surfaces of revolution, when the axis of revolution coincides with the axis of z.

For another example, we will take the equation

$$x\frac{dz}{dx} + y\frac{dz}{dy} = 0.$$

Comparing this to the general equation, we get

$$\text{M} = x \quad \text{and} \quad \text{N} = y,$$

which reduce

$$l(\text{N}dx - \text{M}dy) = du \quad \text{to} \quad l(ydx - xdy) = du;$$

and putting $l = \frac{1}{y^2}$, the integral becomes

$$\int \frac{ydx - xdy}{y^2} = \frac{x}{y} = u,$$

and thence $$z = \phi\frac{x}{y};$$

also

$$m\left(\frac{\text{N}dx - \text{M}dy}{\text{N}}\right) \quad \text{becomes} \quad m\left(\frac{ydx - xdy}{y}\right) = dv,$$

which, by putting $m = \frac{1}{y}$, gives $\frac{x}{y} = v$,

and thence $z = \phi\frac{x}{y}$, the same as before. (See Young's "Differential Calculus," p. 199, &c.)

(4.) We will now show how to integrate equations of the form

$$\text{P}\left(\frac{dz}{dx}\right) + \text{Q}\left(\frac{dz}{dy}\right) + \text{R} = 0,$$

on the supposition that the variables P, Q, R, are functions of x, y, z. Dividing the equation by one of the variables, as by P, and representing the quotients $\frac{Q}{P}$ and $\frac{R}{P}$ by M and N, it becomes

$$\frac{dz}{dx} + M\frac{dz}{dy} + N = 0,$$

or its equivalent

$$p + Mq + N = 0;$$

and from

$$dz = \frac{dz}{dx}dx + \frac{dz}{dy}dy$$

we also have

$$dz = pdx + qdy;$$

consequently, eliminating p, we shall get

$$dz + Ndx = q\,(dy - Mdx),$$

in which q, being clearly arbitrary, we must put

$$dz + Ndx = 0 \quad \text{and} \quad dy - Mdx = 0.$$

If M does not contain z, the equation $dy - Mdx = 0$ admits of a factor m, which makes $m\,(dy - Mdx) = 0$ an exact differential, whose integral gives

$$F\,(x, y) = C = \text{constant}.$$

Hence, if N does not contain z, by eliminating y from $F\,(x, y) = C$, we shall get y in a form that may be expressed by $y = f\,(x, c)$, which, substituted in $dz + Ndx = 0$, will give an integral of the form $z = -\int V dx$, V being a function of x and c; consequently, the indicated integral can be found, whose constant ought clearly, for generality, to be an arbitrary function of the constant C.

Thus, to find the integral of

$$x\frac{dz}{dx} + y\frac{dz}{dy} = a\sqrt{(x^2 + y^2)},$$

by comparing it with the proposed form, we have

$$M = \frac{y}{x} \quad \text{and} \quad N = -a\frac{\sqrt{(x^2 + y^2)}}{x}.$$

Hence the equations

$$dz + N dx = 0 \quad \text{and} \quad dy - M dx = 0$$

will become

$$dz - adx\frac{\sqrt{x^2+y^2}}{x} = 0 \text{ and } dy - \frac{y}{x}\,dx = 0 \text{ or } \frac{xdy - ydx}{x} = 0;$$

and it is clear that the factor $\frac{1}{x}$ reduces the second of these equations to

$$\frac{xdy - ydx}{x^2} = d\,\frac{y}{x} = 0,$$

whose integral gives $\frac{y}{x} = C$ or $y = Cx$.

Consequently, from the value of y in the first of these equations, we shall get

$$dz = adx\,\sqrt{(1 + C^2)},$$

whose integral may clearly be expressed by

$$z = ax\,\sqrt{(1 + C^2)} + \phi C,$$

ϕC being an arbitrary function of C.

Because $C = \frac{y}{x}$, from the substitution of this value, we thence have

$$z = a\sqrt{x^2 + y^2} + \phi\,\frac{y}{x},$$

for the equation between x, y, and z. Resuming the equation

$$dz + N dx = q\,(dy - M dx),$$

on the supposition that the first member does not contain y, and that $dy - M dx$ does not contain z, then, if we have the factors m' and m, which we may have, such that

$$m'(dz + N dx) = du \quad \text{and} \quad m\,(dy - M dx) = dV$$

shall be exact differentials, they will give

$$dz + \mathrm{N}dx = \frac{du}{m'} \quad \text{and} \quad dy - \mathrm{M}dx = \frac{d\mathrm{V}}{m},$$

which will reduce the preceding equation to

$$du = q\,\frac{m'}{m}\,d\mathrm{V};$$

consequently, since the first member of this equation is an exact differential, the second member must also be an exact differential, which it may be (on account of the arbitrariness of q) by putting $q\,\frac{m'}{m}$ equal to a function of V, and thence, by taking the integral, we shall have $u = \phi\mathrm{V}$, or u must be an arbitrary function of V.

Thus, if we take the equation

$$\frac{dz}{dx} + \frac{x}{y}\,\frac{dz}{dy} - \frac{z}{x} = 0,$$

we shall have $\quad \mathrm{M} = \frac{x}{y} \quad$ and $\quad \mathrm{N} = -\frac{z}{x}$,

and thence the equation

$$dz + \mathrm{N}dx = q\,(dy - \mathrm{M}dx)$$

will become

$$dz - \frac{z}{x}\,dx = q\left(dy - \frac{x}{y}\,dx\right) \quad \text{or} \quad \frac{xdz - zdx}{x} = q\left(\frac{ydy - xdx}{y}\right),$$

and thence $m' = \frac{1}{x}$ and $m = 2y$ give

$$\int \frac{xdz - zdx}{x^2} = \frac{z}{x} = u \quad \text{and} \quad \int 2ydy - 2xdx = y^2 - x^2;$$

consequently, we shall have $\frac{z}{x} = \phi\,(y^2 - x^2)$ for the sought integral.

It may be added, that if we eliminate q from the equations

$$p + \mathrm{M}q + \mathrm{N} = 0 \quad \text{and} \quad dz = pdx + qdy,$$

we shall have $\quad Mdz + Ndy = p\,(Mdx - dy)$;

or, since p is arbitrary, as before, we shall get the equations

$$Mdz + Ndy = 0 \quad \text{and} \quad dy - Mdx = 0,$$

and it is clear that we may proceed with these equations in much the same way as before.

Where it may be noticed that we may use the first of these equations $Mdz + Ndy = 0$ instead of $dz + Ndx = 0$ (the first of those before found), since the second equations are identical. If we take the equation

$$\frac{dz}{dx} - \frac{x}{a}\frac{dz}{dy} + \frac{xy}{az} = 0, \quad \text{it gives} \quad M = -\frac{x}{a} \quad \text{and} \quad N = \frac{xy}{az},$$

and these reduce the above equations to

$$2ydy - 2zdz = 0 \quad \text{and} \quad 2ady + 2xdx = 0,$$

whose integrals are $y^2 - z^2$ and $2ay + x^2$; and thence from $u = \phi V$, by putting $y^2 - z^2$ for u, and $2ay + x^2$ for V, we shall have $y^2 - z^2 = \phi\,(2ay + x^2)$. (See p. 50 of vol. ii. of Wright's "Commentary on Newton's Principia.")

(5.) We will here venture some remarks on the integration of the partial differential equations of the second order between x, y, and z, when z is considered as a function of x and y.

1. A partial differential of the second order must involve one of the coefficients

$$\frac{d^2z}{dx^2}, \quad \frac{d^2z}{dy^2}, \quad \frac{d^2z}{dxdy} = \frac{d^2z}{dydx},$$

at the least; and may contain other terms like those that are contained in partial differential equations of the first order.

2. The method of integrating equations of this order is, in some respects, quite analogous to that of integrating partial differential equations of the first order. We will now proceed to integrate some of the simpler forms of equations of the second order.

3. To integrate the forms

$$\frac{d^2z}{dx^2} = 0, \quad \frac{d^2z}{dy^2} = 0, \quad \text{and} \quad \frac{d^2z}{dxdy} = \frac{d^2z}{dydx} = 0.$$

The first of these equations, multiplied by dx, gives

$$\frac{d^2z}{dx} = 0, \quad \text{whose integral is} \quad \frac{dz}{dx} = \phi y;$$

which, multiplied by dx, gives

$$dz = \phi y dx, \quad \text{whose integral is} \quad z = \phi y x + \psi y = x\phi y + \psi y;$$

in which ϕy and ψy represent arbitrary functions of y, which are used instead of the arbitrary constants in common integrations.

By proceeding in like manner to integrate the second of the proposed equations, we shall have

$$\frac{dz}{dy} = \phi x \quad \text{and} \quad z = y\phi x + \psi x,$$

the arbitrary functions being here functions of x.

To integrate the last of the proposed equations under the form

$$\frac{d^2z}{dxdy} = 0, \quad \text{we have} \quad \frac{d^2z}{dx} = 0, \quad \text{and thence} \quad \frac{dz}{dx} = \phi x,$$

whose integral gives $z = \int dx\phi x + \psi y$;

and the integrals of the form $\frac{d^2z}{dydx} = 0$, are

$$\frac{dz}{dy} = \phi y \quad \text{and} \quad z = \int dy\phi y + \psi x.$$

It is manifest, that in this way we shall get

$$\frac{dz}{dx} = \int \mathrm{P}dx + \phi y$$

for the integral of $\frac{d^2z}{dx^2} = \mathrm{P}$,

and $$z = \int \left(\int \mathrm{P}dx + \phi y\right) dx + \psi y$$

is thence the integral of dz, and we have

$$z = \int \left(\phi x + \int \mathrm{P}dy\right) dy + \psi x$$

for the second integral, resulting from $\frac{d^2z}{dy^2} = \mathrm{P}$; and in like manner we have

$$z = \int \left(\phi y + \int \mathrm{P}dx\right) dy + \phi x$$

for the integral resulting from $\frac{d^2z}{dydx} = \mathrm{P}$; noticing, that the equations

$$\frac{d^nz}{dy^n} = \mathrm{P}, \quad \frac{d^nz}{dxdy^{n-1}} = \mathrm{Q}, \quad \frac{d^nz}{dx^2dy^{n-2}} = \mathrm{R}, \text{ \&c.}$$

(P, Q, R, &c., being functions of x and y), may be treated in much the same way.

4. $\frac{d^2z}{dx^2} + \mathrm{P}\frac{dz}{dx} = \mathrm{Q}$, in which P and Q are functions of x and y, can also be easily integrated.

For by putting $\frac{dz}{dx} = u$, the equation becomes

$$\frac{du}{dx} + \mathrm{P}u = \mathrm{Q}, \quad \text{or} \quad du + \mathrm{P}udx = \mathrm{Q}dx,$$

a linear equation, whose integral is expressed by

$$u = e^{-\int \mathrm{P}dx}\left(\int \mathrm{Q}e^{\int \mathrm{P}dx}dx + \phi y\right);$$

and since $u = \frac{dz}{dx}$, we thence readily get

$$z = \int u dx = \int \left\{ e^{-\int P dx} dx \left(\int Q e^{\int P dx} dx \right) + \phi y \right\} dx + \psi y.$$

(See page 455.) It may be added, that the equation

$$\frac{d^2 z}{dx dy} + P \frac{dz}{dx} = Q,$$

by putting $\frac{dz}{dx} = u$, becomes

$$\frac{du}{dy} + Pu = Q, \quad \text{or} \quad du + Pudy = Qdy;$$

whose integral, as before, is

$$z = \int u dx = \int \left\{ e^{-\int P dy} \left(\int Q e^{\int P dy} dy \right) + \phi x \right\} dx + \psi y.$$

In much the same way we can change

$$\frac{d^2 z}{dx dy} + P \frac{dz}{dy} = Q \quad \text{into} \quad \frac{d^2 z}{dy dx} + P \frac{dz}{dy} = Q,$$

since

$$\frac{d^2 z}{dx dy} = \frac{d^2 z}{dy dx}$$

(see p. 22); consequently, putting $u = \frac{dz}{dy}$ we shall have the equation $\frac{du}{dx} + Pu = Q$, or $du + Pudx = Qdx$,

and thence

$$z = \int u dy = \int \left\{ e^{-\int P dx} \left(\int Q e^{\int P dx} dx \right) + \phi y \right\} dy + \psi x.$$

5. It is manifest, from the elimination of $f(ax + by)$ from $X = f(ax + by)$, at p. 26, which gives the equation

$$a \frac{dX}{dy} - b \frac{dX}{dx} = 0,$$

an equation of partial differential coefficients, that equations of partial differential coefficients of the first order must result from the elimination of arbitrary functions from equations, in a way very analogous to that in which ordinary differential equations result from the elimination of arbitrary constants from equations.

Hence, it is clear that in finding the integrals of partial differential equations, we ought analogically to add arbitrary functions to correct the integrals, instead of using arbitrary constants for that purpose; noticing, that the forms of the arbitrary functions must, in particular cases, be determined from the nature of the question.

Thus, if we take the partial differential equation

$$\frac{adz}{dx} + \frac{bdz}{dy} = 1,$$

to find its integral, we may proceed as follows:—

Representing $\frac{dz}{dx}$ and $\frac{dz}{dy}$, as usual, by p and q, the proposed equation becomes $ap + bq = 1$; and since z is a function of x and y, we also have

$$dz = \frac{dz}{dx}dx + \frac{dz}{dy}dy = pdx + qdy.$$

Hence, by eliminating p from the equations

$$ap + bq = 1 \quad \text{and} \quad pdx + qdy = dz,$$

we get $$q\,(bdx - ady) = dx - adz;$$

which, on account of the arbitrariness of q, is equivalent to

$$bdx - ady = 0 \quad \text{and } dx - adz = 0;$$

or eliminating dx from the first of these equations by means of the second, we have

$$dy - bdz = 0 \quad \text{and} \quad dx - adz = 0,$$

whose integrals will clearly be of the form

$$y - bz = A \quad \text{and} \quad x - az = B;$$

consequently, since these equations must clearly be coexistent, we must have $A = \phi B$, or by substituting the values of A and B, we shall have

$$y - bz = \phi(x - az),$$

for the sought integral; indeed, if (as at p. 26) we eliminate the arbitrary function denoted by ϕ from it, we shall get the proposed equation $a\frac{dz}{dx} + b\frac{dz}{dy} = 1$ from it; noticing, that from what is done at p. 211,

$$y - bz = \phi(x - az)$$

is plainly the general form of the equation of cylindrical surfaces, in which the nature of the directrix is undetermined. If, however, the equation of the directrix on the plane x, y, is of the known form $y = fx$, then the nature of the function ϕ can easily be found.

For by putting naught for z in $y - bz = \phi(x - az)$, it becomes $y = \phi x$; consequently, since $y = fx$ we have $\phi x = fx$, which gives the form of ϕ, and of course the equation $y - bz = \phi(x - az)$ will become of the known form

$$y - bz = f(x - az).$$

For further illustration, we will show how to find the integral of $\frac{dz}{dx}x + \frac{dz}{dy} = px + q = 0$.

Since $$dz = pdx + qdy,$$

by substituting $q = -px$, from the preceding equation, for q in it, we shall get $dz = p(dx - xdy)$;

which, on account of the arbitrariness of p, gives

$$dz = 0 \quad \text{and} \quad xdy - dx = 0;$$

or, multiplying by $\frac{1}{x^2}$, $\frac{dy}{x} - \frac{dx}{x^2} = 0.$

By taking the integrals of these differentials, we have

$$z = a \quad \text{and} \quad \frac{y}{x} = b;$$

consequently, since these integrals are coexistent, we must have $a = \phi b$; or, substituting the values of a and b, we shall have $z = \phi \frac{y}{x}$, which, if we please, may be written in the form $\phi^{-1}z = \frac{y}{x}$; which belongs to what are called conoidal surfaces, whose right directrix coincides with the axis of z, without reference to the nature of its curvilinear directrix.

If the curvilinear directrix is given, together with the position of the axis of the conoid, then, putting $z = u$, we shall, from the equations of the curve of double curvature, which represent the curvilinear directrix and the axis, find x, y, and z, in terms of u; consequently, having found x and y in terms of z, we shall get $\frac{y}{x}$ in a function of z, and shall thence get $\phi^{-1}z$ in a known form, which will give ϕ in $z = \phi \frac{y}{x}$, as required.

Again, resuming $z = ax + \phi y$, from p. 583, which is the integral of the partial differential equation $\frac{dz}{dx} = a$; then, it is plain that there is nothing in the nature of the question to determine the form of the function denoted by ϕ in ϕy, so that by putting $x = 0$ we have $z = \phi y$ for the equation of the section of the proposed curve surface, by the plane z, y, between x, y, and z, of an entirely undetermined form.

It is also clear from $z = ax + \phi y$, that the surface, when cut by planes parallel to that of the axes of x and z, always gives right lines which are parallel to each other, since a denotes the tangent of the angle which each of the lines of section makes with the axis of x.

6. It is manifest from what has been done, that the integral of a partial differential equation of the second order between x, y, and z, must involve two arbitrary functions, through which the surface represented by the integral must pass.

Thus (at p. 592), we have found $z = x\phi y + \psi y$ for the integral of $\frac{d^2z}{dx^2} = 0$, in which y and ψy are the arbitrary functions.

If we put $x = 0$, the equation $z = x\phi y + \psi y$ becomes $z = \psi y$, which represents the section of the curve surface by the plane of the axes z and y.

Since the axes of the co-ordinates are supposed to be rectangular, it is clear that ϕy represents the tangent of the angle which the line of common section of the surface by a plane parallel to the axes of x and z makes with the axis of x.

Hence, if a line is drawn in the plane of the section through the point where it cuts the curve $z = \psi y$, supposed to be drawn, at will, to make an angle with the axis of x, having ϕy equal to its (natural) tangent, the line thus drawn will represent the common section of the plane and the surface whose equation is $z = x\phi y + \psi y$, and thence we may readily perceive how the curve surface may be supposed to be described geometrically.

7. It may be added, in concluding this treatise, that the integral of a differential equation containing any number of variables, whether they are total or partial, may clearly be found by Maclaurin's theorem, as explained in (b') given at p. 25.

8. Sometimes the generating function of the integral thus found can be obtained, and thence the integral will be expressed in finite terms.

Thus, if we have

$$z = Z + \frac{dz}{dx}x + \frac{d^2z}{dx^2}\frac{x^2}{1.2} + \frac{d^3z}{dx^3}\frac{x^3}{1.2.3} +, \&c.,$$

in which z is expressed in terms of x, supposing it to be a function of x and y regarded as being independent variables, and $z, \frac{dz}{dx}, \frac{d^2z}{dx^2}$, &c., are supposed to be the values of $z, \frac{dz}{dx}, \frac{d^2z}{dx^2}$, &c., when x is put equal to naught in them; noticing, that if $x = 0$ makes any of the quantities $z, \frac{dz}{dx}, \frac{d^2z}{dx^2}$, &c., infinite, then, by putting $x + a$ for x, we may proceed, as before, to find the expansion according to the ascending powers of x. It is hence clear that the preceding series may be regarded as an integral of a partial differential equation between x and z, in which x and y (or, indeed, any number of variables) are independent variables, and z is a function of them, or depends on them.

If the preceding series has been obtained from a partial differential coefficient of the first order, it is clear that z will represent an arbitrary function of the variables supposed constant in the differential coefficient, and if the series has been obtained from a differential coefficient of the second order, it is plain that z and $\frac{dz}{dx}$ will each be arbitrary functions of the variables supposed to be constant in the differential coefficient, and so on, to any extent that may be required.

9. If we take $\frac{d^2z}{dx^2} = c^2\frac{d^2z}{dy^2}$, it is manifest that we may find z in a series after the following manner.

Thus, by taking the successive differential coefficients, we have

$$\frac{d^3z}{dx^3} = c^2\frac{d^3z}{dy^2dx} = c^2\frac{d^3z}{dxdy^2} = c^2\frac{d^2\frac{dz}{dx}}{dy^2}$$

and

$$\frac{d^4z}{dx^4} = c^2\frac{d^4z}{dx^2dy^2} = c^2\frac{d^2\frac{d^2z}{dx^2}}{dy^2} = c^4\frac{d^4z}{dy^4},$$

since $\frac{d^2z}{dx^2} = c^2\frac{d^2z}{dy^2}$, and we have

$$\frac{d^5z}{dx^5} = c^2\frac{d^5z}{dx^3dy^2} = c^2\frac{d^2\frac{d^3z}{dx^3}}{dy^2} = c^4\frac{d^4\frac{dz}{dx}}{dy^4},$$

and so on, to any extent.

Hence, using ϕy to represent the value of z that corresponds to $x = 0$, $\frac{d^2z}{dy^2}$ corresponding will be expressed by $\phi'' y$, and using ψy to stand for the corresponding value of $\frac{dz}{dx}$, and so on, it is manifest that, from the substitution of the preceding values in the series, we shall get

$$z = \phi y + \psi y\frac{x}{1} + c^2\phi'' y\frac{x^2}{1.2} + c^2\psi'' y\frac{x^3}{1.2.3} + c^4\phi^{\text{IV}} y\frac{x^4}{1.2.3.4} + c^4\psi^{\text{IV}} y\frac{x^5}{1.2.3.4.5} +, \&c.,$$

for the integral, or the required expansion.

If in this series we put $\phi y + \psi y$ for ϕy, and $c(\phi' y - \psi' y)$ for ψy, it will be reduced to

$$z = \phi y + \frac{cx}{1}\phi' y + \frac{c^2x^2}{1.2}\phi'' y + \frac{c^3x^3}{1.2.3}\phi''' y + \&c. +$$

$$\psi y - \frac{cx}{1}\psi' y + \frac{c^2x^2}{1.2}\psi'' y - \frac{c^3x^3}{1.2.3}\psi''' y + \&c.$$

$$= \phi(y + cx) + \psi(y - cx),$$

which expresses the integral of the proposed partial differential equation of the second order, which is the well-known formula for vibrating chords. (See Lacroix, vol. ii., p. 639, and Monge, "Application de l'Analyse à la Géométrie," p. 415.)

To find the total integral of z we may put z, Z, $\frac{dz}{dx}$, $\frac{d^2z}{dx^2}$, &c., for X, (X), $\left(\frac{dX}{dx}\right)$, $\left(\frac{d^2X}{dx^2}\right)$, &c.; $\frac{dz}{dy}$, $\frac{d^2z}{dxdy}$, $\frac{d^2z}{dy^2}$, &c., and so on, for $\left(\frac{dX}{dy}\right)$, $\left(\frac{d^2X}{dxdy}\right)$, $\left(\frac{d^2X}{dy^2}\right)$, &c., and so on, in ($b'$) at p. 25; noticing, that we may put $x + a$, $y + b$, &c., for x, y, &c., and that in this way the integral with reference to all the independent variables, or any number of them, can be found at will.

APPENDIX.

To complete the work, we add the following important articles :—

I.

To what are often called the singular points of curves, we add the following from Todhunter's "Treatise on the Differential Calculus." (See p. 325, &c., of that work.)

1. *Points d'arrêt* are those points of a curve at which a single branch of it suddenly stops.

Thus, $$y = x \log x = \log x \div \frac{1}{x}$$

gives $$y = \frac{dx}{x} \div -\frac{dx}{x^2} = 1 \div \frac{1}{x} = x;$$

which shows that $y = 0$ when $x = 0$, or the curve stops when $x = 0$, which is hence a *point d'arrêt;* but if x is negative, then, since $\log x$ is impossible, it follows that y must be impossible. For the first part of what has been done, see p. 57.

2. A *point saillant* is a point at which two branches of a curve meet and stop, without having a common tangent.

Thus, let $y = \dfrac{x}{1 + e^{\frac{1}{x}}}$, which gives

$$\frac{dy}{dx} = \frac{1}{1 + e^{\frac{1}{x}}} + \frac{e^{\frac{1}{x}}}{x\left(1 + e^{\frac{1}{x}}\right)^2},$$

in which e denotes the base of hyperbolic logarithms.

If x is unlimitedly small, then y in the proposed equation is unlimitedly small also, for two reasons: first, on account of the smallness of x; and second, on account of the unlimitedly great value of $\frac{1}{x}$ in the denominator $1 + e^{\frac{1}{x}}$; and it is clear that the curve touches the axis of x at the origin of the co-ordinates, where $y = 0$. Again, if x is negative, it is easy to see that x unlimitedly small gives y unlimitedly small at the origin of the co-ordinates, or where $x = 0$; and it is also clear that when $x = 0$, we shall reduce $\frac{dy}{dx}$ to

$$\frac{dy}{dx} = \frac{1}{1 + e^{-\frac{1}{x}}} + \&c. = \frac{1}{1 + e^{\frac{1}{x}}} + \&c. = 1,$$

so that $\frac{dy}{dx}$ is the tangent of an arc of 45°; and of course the second branch of the curve lies on the negative side of the axis of x, and makes an angle with x negative of 45°, or half a right angle, and intersects the preceding branch of the curve at the origin of the co-ordinates, making an angle of 135° with it.

3. If a curve has an infinite number of conjugate points, that series of points is called a *branche pointillée*. Thus, if $y^2 = x \sin^2 x$, or $y = \sin x \sqrt{x}$, then, if $x = \pi$ in any integral multiple of π, we shall, for all such points, have $y = 0$, as required.

Remarks.—Since these points do not have place in algebraic curves, yet, since they may sometimes occur in transcendental curves, we have deemed it right to give an account of them.

II.

We here propose to investigate the path that ought to be described by a boat in crossing a river of given breadth, from a given point on one side to a given point on the other, so as to make the passage in the least time possible; supposing the simple velocity of the boat by the propelling power to be given, and that the velocity of the current, being in the same direction with the parallel sides of the river, is variable, and expressed by any given function of the perpendicular distance from that side of the river from which the boat sets out.

It is manifest that the boat, by the propelling power alone, will describe a certain line, either straight or curved, passing from her point of departure to the other side of the river, which is such that the current will float her down the river into another curve, which is formed by the composition of the velocity of the boat in the direction of the first curve and of the velocity of the current, and that the curve thus described, from the point of departure to the point of arrival, will be described in the same time that the propelling power alone would cause her to describe the first curve mentioned, which time, by the question, is to be a minimum.

Let then y, y', denote corresponding ordinates of the two curves (y belonging to the first curve), having x for their common abscissa, the origin of the co-ordinates being at the point of departure, the perpendicular width of the river being the line of the abscissas, and its side the line of the ordinates.

Let V denote the given velocity of the propelling power, and t the time elapsed from the instant of departure; also, let ϕx vary as the velocity of the current at the distance x, and let $a\phi x$, a being a constant, and for simplicity put $a\phi x = x'$. Now, we have

$$\sqrt{dx^2 + dy^2} = \mathrm{V}dt, \quad \therefore\ dy^2 = \mathrm{V}^2dt^2 - dx^2,$$

also $\qquad dy' - dy = x'dt \quad \text{for} \quad dy' - dy$

manifestly denotes the infinitely small distance through which the current floats the boat in the time dt,

$$\therefore dy^2 = (x'dt - dy')^2 = x'^2dt^2 - 2x'dtdy' + dy'^2 = V^2dt^2 - dx^2,$$

since $$dy^2 = V^2dt^2 - dx^2,$$

and thence $$dt^2 + \frac{2x'dy'}{V^2 - x'^2}\,dt = \frac{dx^2 + dy'^2}{V^2 - x'^2}.$$

Solving this quadratic, we have

$$dt = \frac{\sqrt{V^2dy'^2 + (V^2 - x'^2)\,dx^2} - x'dy'}{V^2 - x'^2};$$

which, by putting $\dfrac{dy'}{dx} = p$, taking the integral, &c., gives

$$t = \int dx\left(\frac{\sqrt{V^2p^2 + V^2 - x'^2} - x'p}{V^2 - x'^2}\right),$$

which is to be taken from the given time of departure to the given point of arrival. Since t is to be a minimum, its equal

$$\int dx\left(\frac{\sqrt{V^2p^2 + V^2 - x'^2} - x'p}{V^2 - x'^2}\right)$$

must be a minimum also.

Since, by regarding dx as being constant (because x' is a function of x only), we must, in taking the variations, consider x' as being constant, and take the variations by regarding p only as variable; which gives

$$\delta\int dx\left(\frac{\sqrt{V^2p^2 + V^2 - x'^2} - x'p}{V^2 - x'^2}\right)$$

$$= \int\frac{dx\delta p}{V^2 - x'^2}\left(\frac{V^2p}{(V^2p^2 + V^2 - x'^2)^{\frac{1}{2}}} - x'\right)$$

$$= \int\frac{d\delta y'}{V^2 - x'^2}\left(\frac{V^2p}{\sqrt{V^2p^2 + V^2 - x'^2}} - x'\right),$$

since $$dx\delta p = \delta dy' = d\delta y',$$

by the principles of the Calculus of Variations. Integrating by parts, we have

$$\phi'\delta y''' - \phi\delta y'' - \int \delta y' d\left(\frac{1}{V^2 - x'^2}\right) \times$$
$$\left(\frac{V^2 p}{\sqrt{V^2p^2 + V^2 - x'^2}} - x'\right) = 0,$$

by the nature of maxima and minima. In which ϕ' and ϕ correspond to the first and last points of the curve, which being given, $\delta y'''$ and $\delta y''$, their multipliers, must be equal to naught; and of course the preceding integral is reduced

to $$-\int \delta y' d\left(\frac{1}{V^2 - x'^2}\right)\left(\frac{V^2 p}{(V^2p^2 + V^2 - x'^2)^{\frac{1}{2}}} - x'\right) = 0,$$

which clearly can not be satisfied so as to leave $\delta y'$ arbitrary, except by putting its factor equal to naught, which gives

$$d\left\{\frac{1}{V^2 - x'^2}\left(\frac{V^2 p}{\sqrt{V^2p^2 + V^2 - x'^2}} - x'\right)\right\} = 0,$$

whose integral gives

$$\frac{V^2 p}{\sqrt{V^2p^2 + V^2 - x'^2}} - x' = \frac{V^2 - x'^2}{C},$$

by using $\frac{1}{C}$ for the arbitrary constant.

This equation is clearly equivalent to the form

$$CV^4p^2 - Cx'\sqrt{V^2p^2 + V^2 - x'^2} = (V^2 - x'^2)\sqrt{V^2p^2 + V^2 - x'^2},$$

or we shall have

$$CV^2 p = (V^2 + Cx' - x'^2)\sqrt{V^2p^2 + V^2 - x'^2},$$

or $$C^2V^4p^2 = (V^2 + Cx' - x'^2)^2 (V^2p^2 + V^2 - x'^2),$$

which gives

$$[C^2V^4 - (V^2 + Cx' - x'^2)^2 V^2]\, p^2 =$$

$$(V^2 + Cx' - x'^2)^2 \times (V^2 - x'^2),$$

which gives

$$p = \frac{dy'}{dx} = \frac{(V^2 + Cx' - x'^2)\sqrt{V'^2 - x'^2}}{(V^2 + CV + Cx' - x'^2)^{\frac{1}{2}}(CV - V^2 - Cx' + x'^2)^{\frac{1}{2}}} \div V,$$

or we have

$$V\frac{dy'}{dx} = \frac{(V^2 + Cx' - x'^2)\sqrt{V^2 - x'^2}}{[C^2V^2 - (V^2 + Cx' - x'^2)^2]^{\frac{1}{2}}};$$

whose integral will give the curve described by the propelling power and the action of the water upon the boat during its motion.

To make $\frac{dy'}{dx}$ in the preceding question real, the expression in its denominator positive, so that the square root can be taken, and thence give a real result.

If we omit the terms in the same expression that contain x' and its powers, and put $\frac{dy}{dx}$ for $\frac{dy'}{dx}$, we shall, by a simple reduction, get $\frac{dy}{dx} = \frac{V}{\sqrt{C^2 - 1}}$, for the line described by propelling power alone, from which the current may be supposed to float the boat down into the actual curve described, has the preceding for its differential equation. Because V and C are invariable, it is clear that the integral of the preceding equation is $y = \frac{Vx}{\sqrt{C^2 - 1}}$, which needs no correction, supposing the origin of the co-ordinates to be at the place of departure of the boat.

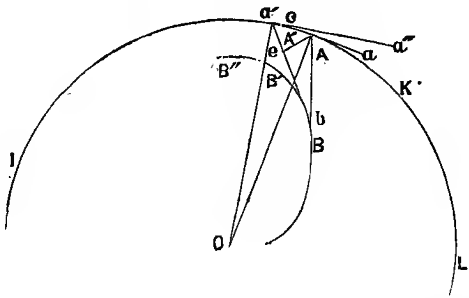
Hence, the propelling power alone causes the boat to describe a right line passing through the given place of departure. To get C, we must obtain the integral of the preceding equation, in which, by putting for y' its value at the given point of arrival, noticing that the correction may clearly be supposed to equal naught, we shall have an equation whose only unknown quantity will be C, which solved gives C; and thence, by taking those values that are not less than 1, the first and last points of the right line described by the propelling power alone become known, and thence the direction or directions, according as C has one or more values, will be found, and the problem solved, as required.

REMARKS.—The question here solved was proposed in No. 2 of the "Mathematical Diary," in the year 1825, by its much accomplished editor and profound mathematician, Robert Adrain, LL. D., then professor of mathematics in Columbia College, New York. I communicated a solution to the question in No. 3 of the same work, which received the prize awarded the solution by the editor. Since there were many mistakes in the published solution, I have concluded, at the earnest solicitation of a former pupil and a much accomplished scientific gentleman, to insert the correct solution of the question in this work.

III.

To illustrate what has been done, suppose the body A moves uniformly around the circumference of the circle LKI with a velocity represented by 1, or unity, while the body B, in pursuit of A, moves continually directly toward A in the curve BB′B″ with the uniform velocity m; then it is proposed to show how to find the nature of the curve described by B, or of the curve of pursuit.

Let AA′ and BB′ be very small parts of the curves described by A and B in the same time, and they will clearly have the ratio 1 : m. Let O be the center of the circle connected with the extremities of the arc AA′ by



the radii AO, A′O, at whose extremities the tangents Aa and A′a'' are drawn, crossing each other in C; then it is evident we shall have the angle aC′A′, made by the tangents, equal to the angle A′OA, subtended by the arc at the center of the circle. Now the angle

$$a\text{AB} = \phi = \text{A}a'b + \text{A}'b\text{A},$$

AB and A′B′ being the corresponding tangents of the curve of pursuit which intersect in b, and thence we have

$$\text{A}'b\text{A} = a\text{AB} - \text{A}a'b = \phi - \text{A}a'b$$
$$= \phi - (\text{CA}'b - \text{A}'\text{C}a') = -\,d\phi + \text{AOA}'$$

by the nature of a differential, since the angle a''A′B′ (supposed to decrease) is successive to aAB $= \phi$. Putting AO $= r$ and the arc AA′ $= dx$, we have

$$\text{A}'b\text{A} = -\,d\phi + \frac{dx}{r};$$

and thence, from the triangle,

$$A'bA \sin\left(\frac{dx}{r} - d\phi\right) =$$

$$\frac{dx}{r} - d\phi \;:\; dx :: \sin AA'b \quad \text{or} \quad \phi \text{ (ultimately)} : Ab = t,$$

which gives $$\frac{t}{r}\, dx - td\phi = \sin \phi dx.$$

Drawing Ae perpendicular to $A'b$, we have $A'e = \cos \phi dx$, and ultimately $eb = Ab$, or $eB'B = AbB$; and thence

$$-dt = B'B - A'e = -\cos \phi dx + mdx,$$

which gives $$dx = -\frac{dt}{m - \cos \phi}.$$

From the substitution of the value of dx, the preceding equation reduces to

$$-\frac{tdt}{r} \div (m - \cos \phi) - td\phi = -\sin \phi dt \div (m - \cos \phi),$$

or $$tdt = r\,[d(\sin \phi, t) - mtd\phi].$$

To integrate this equation, we may clearly assume

$$t = A\phi + B\phi^3 + C\phi^5 +, \&c.,$$

which gives

$$tdt = \frac{d\,[A\phi + B\phi^3 + \&c.]^2}{2}$$

$$= A^2\phi d\phi + 4AB\phi^3 d\phi + 3\,(2AC + B^2)\,\phi^5 d\phi +, \&c.;$$

also from

$$\sin \phi = \phi - \frac{\phi^3}{1.2.3} + \frac{\phi^5}{1.2.3.4.5} -, \&c.,$$

we easily get $$d(\sin \phi, t) =$$

$$2A\phi d\phi + \left(-\frac{2}{3}A + 4B\right)\phi^3 d\phi + \left(\frac{A}{120} - \frac{B}{6} + C\right) 6\phi^5 d\phi +, \&c.,$$

and $$-mtd\phi = -mA\phi d\phi - mB\phi^3 d\phi - mC\phi^5 d\phi -, \&c.$$

Hence, from substitution and omitting the factor $d\phi$, the equation $$tdt = r\,[d\,(\sin\phi,\ t) - mtd\phi]$$
becomes the identical equation

$$A^2\phi + 4AB\phi^3 + 3\,(2AC + B^2)\,\phi^5 + \&c. =$$
$$(2A - mA)\,r\phi + \left(-\frac{2}{3}A + 4B - mB\right) r\phi^3 +$$
$$\left(\frac{A}{20} - B + 6C - mC\right) x\phi^5 +, \&c.\,;$$

which, by equating the coefficients of like powers of ϕ, gives

$$A = (2 - m)\,r, \quad 4AB = \left(-\frac{2}{3}A + 4B - mB\right) r,$$

or $$(4 - 3m)\,B = -\frac{2}{3}A,$$

which gives $$B = -\frac{2}{3}\left(\frac{2 - m}{4 - 3m}\right) r,$$

and $$3\,(2AC + B^2) = \left(\frac{A}{20} - B + 6C - mC\right) r\,;$$

or we have

$$6AC - 6Cr + mCr = \left(\frac{A}{20} - B\right) r - 3B^2$$
$$= \frac{(2 - m)\,(52 - 9m)}{60\,(4 - 3m)}\,r^2 - 3B^2,$$

or $$(6 - 5m)\,C = \frac{(2 - m)\,(52 - 9m)}{60\,(4 - 3m)}\,r - \frac{4}{3}\,\frac{(2 - m)^2}{(4 - 3m)^2}\,r$$
$$= \frac{(2-m)\,(4-3m)\,(52 - 9m) - 80\,(2-m)^2}{60\,(4 - 3m)^2},$$

which gives

$$C = \frac{(2 - m)\,(4 - 3m)\,(52 - 9m) - 80\,(2 - m)^2}{60\,(4 - 3m)^2\,(6 - 5m)}\,r,$$

and so on. Hence,

$$t = C' + \left\{(2 - m)\phi - \frac{2}{3}\left(\frac{2 - m}{4 - 3m}\right)\phi^3 + \frac{(2 - m)\,4 - 3m)\,(52 - 9m) - 80\,(2 - m)^2}{60\,(4 - 3m)^2\,(6 - 5m)}\phi^5 + \&c.\right\} r$$

is the integral, C′ being the constant; and if T = C′ is the value of t at the origin, when $\phi = \phi'$, we shall clearly have

$$t = T + \left\{(2 - m)(\phi - \phi') - \frac{2}{3}\left(\frac{2 - m}{4 - 3m}\right)(\phi - \phi')^3 + \frac{(2 - m)(4 - 3m)(52 - 9m) - 80\,(2 - m)^2}{60\,(4 - 3m)^2\,(6 - 5m)}(\phi - \phi')^5 + \&c.\right\} r$$

for the correct integral.

To find x, we take the equation $dx = -\dfrac{dt}{m - \cos\phi}$; then, from the value of t, we get the form

$$-dt = -[A d\phi + 3B(\phi - \phi')^2 d\phi + 5C(\phi - \phi')^4 d\phi + \&c.];$$

consequently, since

$$\cos\phi = 1 - \frac{\phi^2}{1.2} + \frac{\phi^4}{1.2.3.4} -, \&c.,$$

by putting $m - 1 = m'$, we get

$$dx = -\left\{\frac{A}{m'}d\phi + \frac{6m'B - A}{2m'^2}(\phi - \phi')^2 d\phi + \frac{(m + 6)A - 36m'B + 120m^2C}{24m'^3}(\phi - \phi')^4 d\phi + \&c.\right\} r,$$

whose integral gives

$$x = -\left\{\frac{A}{m'}(\phi - \phi') + \frac{6m'B - A}{6m'^2}(\phi - \phi')^3 + \frac{(m' + 6)A - 36m'B + 120m'^2C}{120m'^3}(\phi + \phi')^5 + \&c.\right\} r,$$

which needs no correction, supposing it commences with

$\phi = \phi'$, or to equal naught at the origin of the motion. If in this value of x we put

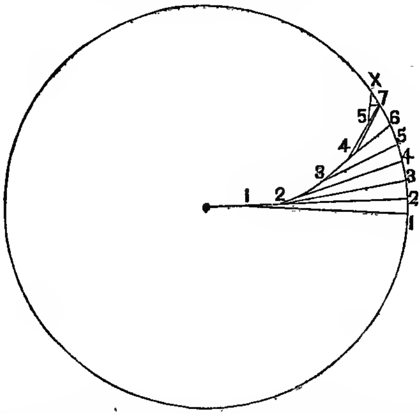
$$A = 2 - m, \quad B = -\frac{2}{3}\left(\frac{2-m}{4-3m}\right), \text{ \&c.,}$$

as in t, we shall get the required value of x.

By taking $\phi - \phi'$ sufficiently small, we can, from the formulas found, find the corresponding values of t and x; and then, changing ϕ into ϕ', and putting $\phi - \phi'$ for a new value of $\phi - \phi'$, we may calculate the corresponding values of t and x as before, and so on, to any required extent; consequently, in this way we may find any number of points in the required curve of pursuit.

REMARK.—This example is given to illustrate the method of integration given by the series in Remark 1, at p. 555.

On account of the complication of the preceding process, we will insert a more simple method by linear description.



Thus, let A start from the position 1 in the circumference of the circle, while B starts from 0 within the circle, and let the bodies move uniformly over the distances 1, 2; 2, 3;

3, 4, &c., and over the distances 0, 1; 1, 2; 2, 3, &c., such, that the first distances being each represented by 1, the second distances (01, &c.) shall each be represented by m. Then will the curve described by B, the pursuing body, be represented by the rectilineal figure 0, 1, 2, 3, &c., nearly, and thence the curve of pursuit can be approximately found; and it is evident that the figure can be described in a slightly different manner, which sensibly gives the same result as before.

IV.

Having procured the last work on Quaternions by the late Sir William Rowan Hamilton, LL. D., M. R. I. A., &c. published in 1866, since the much lamented death of the gifted author, by his son William Edwin Hamilton, and having given much time to the study of the work, we here propose to notice some parts of the work that are somewhat analogous to what has been done in the preceding treatise.

To the end in view, we shall refer to what is done by the author at p. 215 of his treatise, as follows:

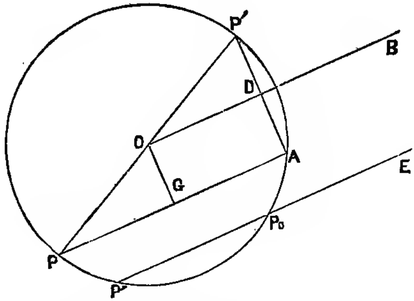
(1.) From a point A of a sphere with O for center, let it be required to draw a chord AP, which shall be parallel to a given line OB, or more fully, *to assign the vector*, $\rho = \text{OP}$, of the extremity of the chord so drawn, *as a function of the two given vectors*, $\alpha = \text{OA}$ and $\beta = \text{OB}$; or rather, of α and Uβ, since it is evident that the length of the line β can not affect the result of the construction, which the figure may serve to illustrate.

(2.) Since AP ‖ OB or $\rho - \alpha \parallel \beta$, we may begin by writing the expression

$$\rho = \alpha + x\beta \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ (1).$$

which may be considered as a form of the *equation of the right line* AP, and in which it remains to determine the scalar coefficient x, so as to satisfy the equation of the sphere

$$T\rho = T\alpha \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2).$$



In short, we are to seek to satisfy the equation

$$T\,(\alpha + x\beta) = T\alpha$$

by some scalar x which shall be in general different from zero, and then to substitute this scalar in the expression $\rho = \alpha + x\beta$, in order to determine the required vector ρ.

(3.) For this purpose, an obvious process is, after dividing by $T\beta$, to square, and to employ the formula 210, XXI., which had indeed occurred before, as 200, VIII., but not then as a consequence of the distributive property of multiplication. In this way we get

$$\left(\frac{T\alpha}{T\beta}\right)^2 = \left(\frac{T\alpha}{T\beta} + x\right)^2, \quad \text{or} \quad \frac{2\alpha}{\beta}\,x + x^2 = 0,$$

which is satisfied either by

$$x = 0, \quad \text{or} \quad \frac{2\alpha}{\beta} + x = 0,$$

which gives $x = -\dfrac{2\alpha}{\beta}$. Substituting this value for x in the equation $\rho = \alpha + \beta x$, we have $\rho = \alpha - 2\alpha = -\alpha$, on account of the negative sign; consequently, AP is clearly parallel to OB as required, or, as in Hamilton,

$$\text{CA} = \frac{\alpha - \rho}{2} = \alpha,$$

and the figure OCAD is a parallelogram.

Instead of this process, we raise the members of the equation $\frac{\alpha}{\beta} = x + \frac{\alpha}{\beta}$ to the integral power n, and retain only those terms that involve the first power of β, and thence, on account of the indefiniteness of β, we get

$$0 = x^n + nx^{n-1}\frac{\alpha}{\beta},$$

which gives $x = -\frac{n\alpha}{\beta}$; consequently, from $\rho = \alpha + x\beta$, we have $\rho = -(n-1)\alpha$, which, for $n = 2$, gives $\rho = -\alpha$, as before. It is evident that we may in like manner take $n = 3, 4, 5$, &c., and thence draw any number of parallels to OB, which will not, however, pass through the point A; and it is evident that, in like manner, a parallel to OB may be drawn through E, as in the figure. Thus a sphere, having its center at O and radius OE, cuts P′P produced toward P′, so that its arc between E and where it meets OP′ produced will be bisected by OB; and of course the right line EP°P′, joining E, and where OP produced meets the sphere, must be parallel to OB.

Remarks.—It is clear that the preceding method of treating the problem will be found useful in all analogous cases. We will add that Hamilton represents a quaternion in its most general form by $q = \omega + ix + jy + kz$, in which the term ω, and the multipliers x, y, z, are called scalars,

$$i^2 = j^2 = k^2 = ijk = -1,$$

i, j, k, being so taken that they shall represent a system of three right versors in three rectangular planes, as described by Hamilton at p. 157, Art. 181; and for the method of using these versors in practice we shall refer to p. 366, &c., of Hamilton, where he will find some well-known formulas of spherical trigonometry.

Finally, we would advise the student to make himself familiar with Hamilton's definitions; to read with care the parts of the work to which reference has been made. Besides he will also do well to read from p. 208, Art. 210, continuously to p. 240, Chapter II. Indeed, whoever will give his attention to obtain a thorough knowledge of the work, will find his labor abundantly rewarded.

THE END.